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Sustainability with Risky Growth

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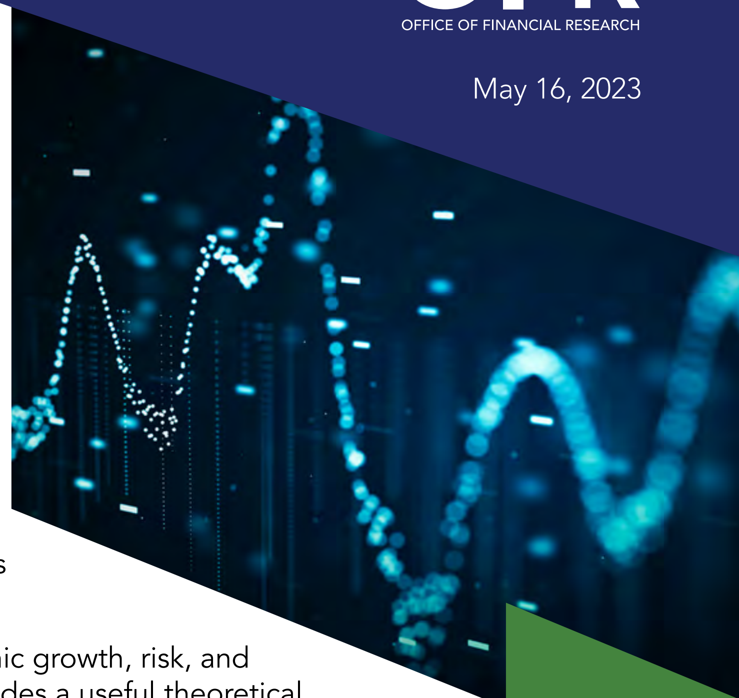

Why this Study is Important

Several studies have focused on how climate risk could affect financial stability. We take a complementary approach by examining how growth, risk, and financial depth contribute to sustainability. Using a stochastic growth model, we explore the conditions under which an economy is sustainable, considering different types of risk.

This research will help policymakers understand how economic growth, risk, and the financial sector influence sustainability objectives. It provides a useful theoretical framework useful to help assess what policies related to growth and financial depth are likely to affect sustainability.

Key Findings

1. In general equilibrium, sustainability is improved by a higher rate of expected GDP growth, a decrease in GDP growth risk, and improved depth of financial markets. The expected rate of GDP growth always has a first-order effect on sustainability, so increasing growth may be the primary way to achieve sustainability objectives.

2. Whether growth risk has first- or second order consequences for growth depends on the nature of risk. For standard variations in growth captured by the volatility of GDP, risk has a second-order impact on sustainability considerations. However, long-run risks to growth, such as persistent decreases in growth rates due to "tipping points" could have first-order consequences.

3. Sustainability concerns could change the optimality of investment decisions when considering long-run growth concerns. Investors may choose high-growth, high-risk technologies that are privately optimal, but not socially optimal, thus providing a motivation for regulation.

How We Did This Study

We proposed a theoretical macro model that could provide a conceptual framework for analyzing how an economy with risky growth achieves sustainability. By our definition, an economy is sustainable if consumer welfare not is expected to decrease over time.

The theoretical model combines optimal investment and this sustainability condition on expected utility into a macro growth model. In contrast to the existing literature that assumes returns are given (partial equilibrium), we considered how general equilibrium determines the returns on safe and risky investments based on underlying growth and risk. We also considered how these factors affect the sustainability condition.

Finally, we incorporated different types of shocks to determine when the sustainability condition is likely to be satisfied.

Sustainability with Risky Growth

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Abstract

We investigate the fundamental determinants of sustainability in general equilibrium. We adopt a definition of sustainability that requires that the welfare of future generations is not expected to decrease on average. We then use a stochastic growth model to explore the conditions under which an economy is sustainable, considering different types of risk. In general equilibrium, sustainability boils down to supply-side factors, with increased growth, decreased consumption risk, and greater financial depth making sustainability more likely. Our results have policy implications regarding endogenous investment, catastrophic-risk management, investments in clean and dirty technologies, and the importance of risk-sharing assets.

Keywords: growth, rare disasters, climate risk, long-run risk, welfare, intergenerational equity, financial depth.

JEL Classification: D61, D63, E22, G11, H12, O40, Q01, Q54

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1 Introduction

In October of 2022, members of the Last Generation group threw mashed potatoes at one of Monet's famous haystack paintings in a museum in Potsdam, Germany. The action was meant to highlight the urgency of addressing climate change, despite the distractions of more immediate challenges such as energy scarcity and global inflation. The protest reflects a tension between the welfare of current generations, who may be more focused on the needs of the present, and the welfare of future generations, who will bear the consequences of delayed action on everything from climate change to infrastructure to national debt. This tension raises fundamental questions about the sustainability of household, firm, and government decisions.

What determines whether current decisions are in some sense sustainable? One approach, pioneered by Arrow et al. (2004) and Campbell and Martin (2023), defines decisions as *sustainable* if they provide for a nondecreasing expected path of future welfare. This notion of sustainability reflects the language of the report by the World Commission on Environment and Development (1987), which defines *sustainable development* as "[meeting] the needs of the present without compromising the ability of future generations to meet their own needs." In this sense, the criterion of sustainability operates as an additional constraint on the choices facing society. This paper investigates the fundamental economic determinants of a sustainability criterion in a general-equilibrium setting, where asset prices, consumption decisions, and economic growth are all interconnected and ultimately driven by macro fundamentals underlying safe and risky rates of return.

Our analysis is motivated and inspired by Campbell and Martin (2023), who provide the seminal definition of sustainability in a risky world and characterize conditions for sustainability in terms of safe and risky returns. Using their definition and methodology, we consider the underlying equilibrium determinants of sustainability by endogenizing returns within a stochastic growth model. By embedding their analysis within a stochastic growth model, we can characterize conditions for sustainability in terms of the underlying fundamental aggregate variables that endogenously determine returns in equilibrium, rather than stating conditions in terms of safe and risky returns. Our analysis provides three broad sets of insights related to the fundamental determinants of sustainability, the nature of aggregate risks, and implications for policy.

First, considering general equilibrium in a stochastic growth model provides new insights into the underlying determinants of sustainability. We show that sustainability is determined by the levels of growth, aggregate risk, and financial depth. Growth is beneficial for sustainability, while risk of all kinds is negative. Perhaps surprisingly, greater financial depth that allows consumers to borrow more easily against the future *improves* sustainability. While one might worry that financial depth would *deplete* future generations of available resources as a result of debt, in equilibrium financial depth allows an economy to grow wealth at a faster rate and thus provide more resources for future generations. The fundamental roles of growth and risk for sustainability in general equilibrium stand in contrast to the determinants of sustainability in partial equilibrium. While partial-equilibrium models highlight the importance of the discount factor in sustainability (see the related literature below), the consumer discount factor has no direct impact on the sustainability condition in general equilibrium.

Second, our analysis provides insights into how risk affects sustainability. While Campbell and Martin (2023) show that in general risk can relax the sustainability constraint, we find that the opposite is true in a stochastic growth setting. In the partial-equilibrium setting, the risk premium ensures that the positive expected drift of wealth more than offsets the negative impact of risk on expected welfare due to risk aversion. In this sense, the presence of risk has a net positive effect on the sustainability of consumption. In general equilibrium, however, risk has very different consequences for sustainability: here, wealth grows at the rate of economic growth, while economic volatility does not yield a risk premium on wealth in the aggregate, but merely depresses the risk-free rate to provide a risk premium in returns. In general equilibrium, all types of risk make it more difficult for an economy to meet its sustainability condition. (For a similar result, see Section 4 of Campbell and Martin, 2023.)

Third, we show that the types of aggregate risk facing an economy matter for sustainability. In the case of Brownian volatility, the risk term has only a second-order impact on sustainability and is unlikely to matter very much. Rare-disaster risk, in contrast, can have a much more significant impact on the sustainability condition. Depending on the arrival rate and severity of the rare event, we find that disaster risk can overwhelm the combined impact of growth and higher-frequency risk. We also consider the effect of long-run nonstationary risks that can lead to persistent decreases in the economic growth rate. We investigate the impact of nonstationary risk in both partial and general equilibrium and show that this type of risk makes sustainability less likely in both settings. (This stands in contrast to the impact of stationary risk, which actually relaxes the sustainability constraint in partial equilibrium, as in Campbell and Martin, 2023.) We consider a particular case of differing long-run risk by introducing "green" and "brown" technologies, where the brown technology generates higher growth at the expense of increased long-run risk. In the absence of a sustainability condition, the economy would tilt toward the higher-growth brown technology. Sustainability considerations that take into account long-run risk, however, would induce increased investments in the green technology.

Finally, our analysis provides novel policy implications for sustainability concerns that go beyond the previous study of savings rates and appropriate discount factors. Though simple, the general-equilibrium condition highlights the fundamental role of supply-side factors in determining the sustainability of an economy, and it points to a different set of policy implications than emerge when considering the underlying determinants of returns, rather than returns directly. We explore some of these policy implications by allowing for endogenous growth, catastrophic-risk mitigation, financial-market depth, and financial-market integration. Growth plays a fundamental role in our general-equilibrium sustainability condition. In order to examine the impact of growth policies, we endogenize growth through a simple AK model that allows investment in physical or human capital to raise growth rates. The message here is straightforward: any policies that improve growth outcomes will push an economy further above its sustainability threshold.

On the risk side of the equation, we examine policies that affect both the frequency and the impact of rare disasters (in the spirit of Barro, 2006), as well as the role of higher-frequency risks captured by our Brownian process. Given the large potential impact of disaster risk on sustainability, we discuss policies that influence both the arrival rate and the magnitude of disasters. Canonical examples of such policies include climate and nuclear treaties, as well as mitigation strategies. Managing higher-frequency risks, however, points to financial solutions, including reducing financial frictions, improving financial depth, and optimizing economic openness. Following Caballero, Farhi, and Gourinchas (2008), for instance, we model financial depth as a fraction of future cash flows that can be capitalized as assets. We find that the higher this fraction, the easier it is for countries to meet the sustainability condition. This effect is dampened, however, in the presence of increased financial integration.

Related Literature The standard approach in economics has focused on whether current decisions maximize the discounted expected present value of current and future utility. As is well known, the optimality of decision-making in this setting depends crucially on the choice of discount rate, with controversy surrounding the ethics of discounting the welfare of future generations at all. See Dasgupta (2008, 2021) for reviews of the literature on discounting. Arguments favoring the use of a low discount rate for the distant future include Dybvig et al. (1996), Weitzman (1998), and Gollier (2002). In the context of climate change, the Stern Report (Stern, 2007) has argued for the use of what is effectively a zero discount rate, which has received criticism from Nordhaus (2007), among others. An alternative approach defines decisions as "sustainable" if they provide for a nondecreasing expected path of future welfare. Arrow et al. (2004) apply this criterion to various regions around the world and find mixed evidence for the sustainability of consumption and investment decisions, with poorer regions showing declining investment in a broad measure of the productive base ("genuine wealth") and rich countries showing positive growth. While there may be other definitions of sustainability, this one is the starting point for our paper.

The condition in Arrow et al. (2004) does not account for capital market risk. Campbell and Martin (2023) introduce risky capital and show that the sustainable social rate of time preference (the consumption-wealth ratio) lies between the risk-free rate and the risky return on capital. Because it earns a risk premium, risky capital provides additional space for sustainable wealth accumulation, allowing the consumption-wealth ratio to rise above the risk-free rate. Campbell and Martin (2023) consider the criterion of sustainability as an additional constraint on the choices facing society. Importantly, in a setting with exogenous rates of return, with a safe but low-returning investment and a risky but high-returning investment, the economy allocates capital correctly, in the sense that the sustainability constraint does not distort the investment decision. In contrast, we show that sustainability concerns can distort the investment decision in an economy with long-run risk.

The impact of risk on the sustainability constraint naturally depends on the specification of the social welfare function. Pindyck (2022) applies the Campbell and Martin (2023) framework to a social welfare function that allows for population to affect utility through its effect on both per-capita consumption and the intrinsic value of the population itself, as well indirectly through its effect on total output. These additional population channels provide for interesting tradeoffs between population growth and sustainable consumption, including the possibility that societies may be willing to tradeoff higher future populations against lower future consumption. Given our focus on the general equilibrium properties of sustainability, we maintain the assumption in Campbell and Martin (2023) and model social welfare as depending only on the future stream of consumption.

The Campbell and Martin (2023) framework primarily operates with exogenous returns. It does, however, consider endogenous returns within a finance setting (i.e., in the case when the risk-free rate is determined endogenously and the sustainability criterion is framed in terms of returns and risk premia) and arrives at a sustainability condition that is equivalent to our baseline case with stationary risk. Given the broad interpretation of Campbell and Martin's sustainability framework, which extends to both the national level and the world as a whole, it's natural to rephrase their exercise in terms of fundamental aggregate variables. As noted, in general equilibrium, asset prices, consumption decisions, and economic growth are all interconnected and ultimately driven by macro fundamentals underlying safe and risky rates of return. This issue motivates our paper.

In studying the general-equilibrium considerations regarding sustainability, we are building on a literature that goes all the way back to Ramsey (1928) and the question of how much an economy should save for the future. The intergenerational-justice implications of such savings decisions were developed in (among others) Rawls (1999), Arrow (1973), Solow (1974), and Hartwick (1977). Dasgupta (2021) provides a comprehensive summary of the philosophical and economic tensions involved in any question of intertemporal welfare. Not surprisingly, some of the thorniest issues center on the question of the appropriate discount rate, with Stern (2007) applying a near-zero discount rate to argue for drastic changes in the present, and others highlighting the implications and hazards of low discount rates (Nordhaus, 2007), the need to balance the costs and benefits of any sustainability strategy (Nordhaus, 1991), and the importance of modeling catastrophic risk (Weitzman, 2007). Rather than imposing sustainability as a constraint, we characterize when in equilibrium an economy is likely to satisfy the sustainability condition.

More directly, a number of studies formalize the ethical requirement that either consumption or welfare should not decrease over time. We have already mentioned Campbell and Martin (2023), Pindyck (2022), and Arrow et al. (2004), which are the closest to our approach. These studies, however, build on a series of contributions that address different aspects of sustainable welfare conditions, including Pezzey (1992), Howarth (1995), Solow (1995), Dietz and Asheim (2012), and Campbell and Sigalov (2022). As is often the case, Solow (1995) provides an especially eloquent case for a sustainability constraint: "The duty imposed by sustainability is to bequeath to posterity not any particular thing [...] but rather to endow them with whatever it takes to achieve a standard of living at least as good as our own and to look after their next generation similarly." Our contribution is to apply this standard in a general-equilibrium setting with different sources of risk and policy levers.

Outline The rest of this paper proceeds as follows. Section 2 compares the sustainability condition in a partial-equilibrium setting to that in a general-equilibrium one, with Brownian volatility.

Section 3 then shows how these results depend on the presence of disaster risk, a persistent Markov regime-switching process, and the presence of clean and dirty technologies. Section 4 presents extensions that consider financial development in greater detail. Section 5 discusses various policy options aimed at relaxing the sustainability constraint, and Section 6 concludes.

2 Sustainability in a Stochastic Growth Model

The notion of "sustainability" has taken on a variety of meanings in the popular press, policy circles, and academic journal articles. Decades ago, the World Commission on Environment and Development (1987) defined sustainability as "[meeting] the needs of the present without compromising the ability of future generations to meet their own needs"—but what does that mean? At the broadest level, it means that the current path of decisions leaves future generations at least as well off as the current generation. In this interpretation, sustainability is less about optimizing current decisions than about satisfying a condition on the trajectory of expected welfare across future generations. This is the approach taken by Arrow et al. (2004) and Campbell and Martin (2023), and it is the starting point for our own analysis.

2.1 Partial Equilibrium

As in Campbell and Martin (2023), we begin by considering a standard continuous-time optimization model, in which a representative investor chooses consumption and a risky portfolio share to maximize the expected discounted value of lifetime utility given exogenous returns. Investment risk takes the form of a Brownian process, an assumption that we will relax in Section 3. We begin with the simplest case of log utility and then extend the results to CRRA and recursive preferences.

A representative investor discounts the future at rate ρ and chooses consumption c_t and the risky portfolio share α_t to maximize expected discounted lifetime utility:

$$\mathbb{E} \int_0^\infty e^{-\rho t} \log c_t \, \mathrm{d}t.$$

There is a risk-free asset with return r and a risky asset with Brownian volatility σ and excess return μ. Wealth W_t therefore follows the process

$$\frac{\mathrm{d}W_t}{W_t} = (r + \alpha_t \mu - \theta_t) \, \mathrm{d}t + \alpha_t \sigma \, \mathrm{d}Z_t,$$

where $\theta_t \equiv c_t/W_t$ is the consumption-wealth ratio and Z_t is a Wiener process. Letting $V(W_t)$ denote the value function, the Hamilton-Jacobi-Bellman (HJB) equation is:

$$\rho V = \max_{c_t,\alpha_t} \left\{ \log(c_t) + V'(r + \alpha_t \mu - \theta_t) W_t + \alpha_t^2 \frac{\sigma^2}{2} W_t^2 V'' \right\}. \tag{1}$$

With log utility, the value function can be written as

$$V(W_t) = A \log W_t + b,$$

where A and b are constants, so that $V'(W_t) = A/W_t$ and $V''(W_t) = -A/W_t^2$.

As is standard, optimal consumption is $c = \rho W$, and the optimal share in the risky asset is $\alpha = \mu/\sigma^2$. Substituting these optimal values into equation (1), we can solve for welfare as:

$$V_t = \frac{\log W_t}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR^2}{2} \right), \tag{2}$$

where $SR \equiv \mu/\sigma$ denotes the Sharpe ratio.

Since welfare only depends on time through wealth, the expected change in welfare is just the expected drift in log wealth divided by the discount rate, ρ. Letting $\nu \equiv \mathbb{E}[\mathrm{d}\log W_t]$ denote the expected drift in log wealth, we have

$$\mathbb{E}[\mathrm{d}V_t] = \frac{\mathbb{E}[\mathrm{d}\log W_t]}{\rho} \equiv \frac{\nu}{\rho}.$$

The definition of sustainability from Campbell and Martin (2023) requires that welfare cannot have a negative drift, which amounts to $\nu \geq 0$. By Itô's Lemma, the evolution of log wealth is

$$\begin{aligned} \mathrm{d}\log W_t &= \frac{\mathrm{d}W_t}{W_t} - \frac{1}{2}\left(\frac{\mathrm{d}W_t}{W_t}\right)^2, \\ &= \left(r - \theta + \alpha\mu - \alpha^2\sigma^2/2\right)\mathrm{d}t + \alpha\sigma\mathrm{d}Z, \end{aligned}$$

since $(\mathrm{d}W_t/W_t)^2 = \alpha^2\sigma^2\mathrm{d}t$. Hence, the expected drift ν is given by

$$\nu = r - \theta + \underbrace{\alpha\mu}_{\substack{\text{return}\\ \text{adjustment}}} - \underbrace{\alpha^2\sigma^2/2}_{\substack{\text{utility}\\ \text{adjustment}}}. \tag{3}$$

It is worth noting how risk affects the expected change in welfare in partial equilibrium. First, risk operates through a positive return adjustment relative to the risk-free rate. Risky investments earn a risk premium, which increases the drift in wealth by $\alpha\mu$ and generates a higher average return. All else equal, a higher risk premium is good for sustainability because it increases the expected trajectory of resources available in the future.

Second, risk operates through a negative utility adjustment. Relative to the drift in wealth, which is $r + \alpha\mu - \theta$, the drift in welfare is lower by the term $\alpha^2\sigma^2/2$. This utility adjustment is due to the curvature in the welfare function (risk aversion). With concave utility, higher risk decreases the expected value of future welfare (Jensen's inequality). This force puts downward pressure on the expected trajectory of welfare. Whether risk overall increases or decreases the drift of welfare depends on whether the return adjustment outweighs the utility adjustment.

At the optimal portfolio share $\alpha = \mu/\sigma^2$, the risk premium term equals SR^2, while the welfare adjustment is $-SR^2/2$. Evaluating the expected drift at the optimal values of consumption and the risky share, we have

$$\nu = r - \rho + \frac{1}{2}SR^2.$$

With exogenous returns, the net impact of risk on the expected welfare trajectory is positive: the risk premium more than offsets the negative impact of risk on welfare. This is one of the main lessons from Campbell and Martin (2023): risk allows for a positive drift in wealth, even with a binding sustainability constraint. From the preceding equation, the agent will choose a sustainable consumption-investment profile if

$$r + \frac{1}{2}SR^2 \geq \rho. \tag{4}$$

Despite the relatively simple setting (log utility and Brownian volatility), we can already see an important dynamic regarding sustainability in partial equilibrium. The drift in welfare increases with the risk-free rate, patience, and the Sharpe ratio.[1]

Importantly, this basic intuition about the partial-equilibrium result holds for different assumptions about preferences and the nature of risk (see Appendix A for derivations). For example, with Epstein-Zin preferences (Epstein and Zin, 1989; Duffie and Epstein, 1992) with elasticity of intertemporal substitution (EIS) ϵ and coefficient of relative risk aversion γ, the agent will choose

[1] When imposed as a constraint as in Campbell and Martin (2023), we can solve for the constrained consumption-wealth ratio as $\theta_{\text{con}} = r + \frac{1}{2}SR^2$, which lies between the risk-free rate and the expected return on optimally invested wealth, $r + SR^2$.

a sustainable consumption-investment profile if

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0, \tag{5}$$

which is independent of ϵ. This condition nests the case of additive CRRA utility ($\epsilon = 1/\gamma$). In either case, the risk term in the sustainability constraint is modified by the level of risk aversion. Considering higher risk aversion than log utility ($\gamma > 1$) tightens the sustainability condition.

All of these conditions share the common property that the presence of risk provides additional "room" for achieving sustainable welfare profiles, in the sense that the discount rate can exceed the risk-free rate. This analysis assumes, however, that returns are exogenous.

2.2 General Equilibrium

How do things change in a general-equilibrium economy, in which asset prices need to be consistent with consumption and saving decisions? We now endogenize safe and risky returns in the simplest setting possible: a Lucas (1978) tree economy with a single risky asset and a risk-free asset in zero net-supply. Section 2.3 extends to a setting with investment and endogenous growth. We again assume that the representative consumer has log preferences.[2]

The tree accounts for all the productive factors in the economy. The dividend stream D_t (the "fruit") follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = g\,\mathrm{d}t + \sigma\,\mathrm{d}Z_t, \tag{6}$$

where g is the growth rate, σ is the volatility of risk, and Z_t is a Weiner process.

The tree has a per-dividend price Q_t (the "asset price"), and the risk-free rate r_t is endogenously determined in equilibrium. In this setting, the asset price Q_t and the risk-free rate r_t need to adjust so that the representative consumer is willing to hold the entire tree and consume the dividends.

Because the tree represents all productive factors in the economy and the risk-free asset is in zero net-supply, aggregate wealth is $W_t = Q_t D_t$. Furthermore, since optimal consumption is

[2] Campbell and Martin (2023) consider general equilibrium with linear technologies (exogenous safe and risky returns) or in a case in which only a risky technology exists. In that case, the risk-free rate is endogenous, but they maintain an exogenous risk premium. One can view our analysis as providing the fundamental macro determinants for the return on the risky investment.

$c_t = \rho W_t$ and consumption equals the tree's dividends ($c_t = D_t$), the asset price is stationary:

$$\rho Q = 1 \implies Q = \frac{1}{\rho}.$$

Since all wealth is invested in the tree, household wealth grows at the rate of economic growth, g.

The consumer's stochastic discount factor, $\Lambda_t \equiv e^{-\rho t}/c_t$, prices all assets (see Cochrane, 2009, 2017). Since consumption equals the tree's dividends, the drift and volatility terms for consumption are the same as those for the dividends. Because consumption is proportional to wealth, the volatility of both consumption and wealth is σ. In equilibrium, the risk premium on the tree equals the volatility of the stochastic discount factor times the volatility of dividends. Hence, the Sharpe ratio of the risky investment equals the volatility of consumption. Since the optimal portfolio allocation sets $\alpha = \mu/\sigma^2$ and $\alpha = 1$ by market clearing, the equilibrium returns on the tree satisfy

$$SR = \sigma \implies \mu = \sigma^2.$$

By the stochastic maximum principle, the risk-free rate satisfies

$$r_t \,\mathrm{d}t = -E_t\left[\frac{\mathrm{d}\Lambda_t}{\Lambda_t}\right] = \rho \,\mathrm{d}t + E_t\left[\frac{\mathrm{d}c_t}{c_t}\right] - Var_t\left[\frac{\mathrm{d}c_t}{c_t}\right], \tag{7}$$

implying a constant risk-free rate given by the familiar equation:

$$r = \rho + g - \sigma^2. \tag{8}$$

While this is a standard result, it is worth understanding the intuition for how risk affects the risk-free rate in equilibrium. The expected return on the tree includes the dividend yield from output, which is $1/Q$, plus expected capital gains due to growth, which is g. Since $Q = 1/\rho$ in equilibrium, the expected total return on investing in the tree is $\rho+g$. The risk premium is $\rho+g-r$. For the tree to earn a risk premium of σ^2, the risk-free rate must adjust so that $\rho + g - r = \sigma^2$, which implies $r = \rho + g - \sigma^2$. Once the risk-free rate adjusts for risk, the expected return on the household's portfolio is simply $\rho+g$. Therefore, after the household consumes at a rate ρ, its wealth grows at an expected rate g.

This has important implications for sustainability. For a given risk-free rate, the sustainability condition with log utility requires that $\nu = r - \rho + \frac{1}{2}\sigma^2 \geq 0$. By substituting the risk-free rate in general equilibrium, $r = \rho + g - \sigma^2$, we get our first main result.

Proposition 1. *With log utility and Brownian risk, an economy satisfies the sustainability criterion in general equilibrium if*

$$g - \frac{1}{2}\sigma^2 \geq 0. \tag{9}$$

Equation (9) is quite different from equation (4), though both reflect the same constraint on welfare. While the household rate of time preference was prominent in partial equilibrium, it is completely absent in the condition in general equilibrium. In addition, risk had the property of *relaxing* the condition in partial equilibrium, while it actually *tightens* the condition in general equilibrium. Risk σ and time-preference ρ are both reflected in the risk-free rate. In general equilibrium, higher risk makes sustainability less likely.

What is the intuition for the contrasting way in which risk operates in general equilibrium? Fundamentally, the drift in log wealth can be decomposed into a growth term and a utility adjustment:

$$\nu = \underbrace{\mathbb{E}[\mathrm{d}W_t/W_t]}_{=g} - \underbrace{\sigma^2/2.}_{\substack{\text{utility}\\ \text{adjustment}}} \tag{10}$$

Recall that in partial equilibrium, risk affected the drift in wealth through two channels: a return channel (the risk premium) and a utility channel (concavity). Fixing the risk-free rate, higher risk led to a higher overall return on the investor's portfolio.

In general equilibrium, the return on household wealth is determined entirely by fundamentals: wealth grows at the rate of economic growth, which is equivalent to the expected growth rate of the tree, g. The risk-free rate adjusts to absorb the risk premium. Thus, the return channel is entirely absent in general equilibrium; higher risk does not lead to higher returns overall because expected returns are determined entirely by the underlying growth rate of the economy. In general equilibrium, risk operates only through the utility adjustment due to concavity. Thus, in general equilibrium, risk makes conditions for sustainability less likely.

As with the partial-equilibrium model, the basic intuition behind the general-equilibrium result does not change with different specifications of preferences (see Appendix A for derivations). For instance, with Epstein-Zin preferences, the risk-free rate satisfies

$$r = \rho + \frac{1}{\epsilon}g - \frac{\gamma\left(1 + \frac{1}{\epsilon}\right)}{2}\sigma^2,$$

where ϵ and γ are again the EIS and coefficient of risk aversion, respectively. The excess risk premium equals $\gamma\sigma^2$, which means $SR = \gamma\sigma$. Substituting these into the sustainability condition, we have

$$\epsilon \left(\rho + \frac{1}{\epsilon} g - \frac{\gamma \left(1 + \frac{1}{\epsilon}\right)}{2} \sigma^2 - \rho + \frac{\gamma^2 \sigma^2}{2\gamma} \right) \geq 0, \tag{11}$$

which reduces to

$$g - \frac{1}{2} \gamma \sigma^2 \geq 0. \tag{12}$$

As before, the ϵ drops out and there is only an adjustment for risk aversion, which (intuitively) raises the bar for meeting the sustainability threshold.

The sustainability requirement in general equilibrium is markedly different from the one in partial equilibrium, and it yields several key insights. First, sustainability in general equilibrium depends only on a combination of economic growth and economic volatility (g and σ). It does not at all depend on the discount rate, ρ. To the extent that the general-equilibrium framework captures the fundamental forces facing the economy, this has important implications for any policies aimed at avoiding a binding sustainability constraint. Policies that favor economic growth and/or result in lower volatility will make it more likely that an economy is sustainable. This places the emphasis on factors that have been central in the literatures on endogenous growth, macro-finance, and environmental sustainability.

Second, for plausible values of growth and Brownian volatility, most economies are likely to satisfy the sustainability requirement. Taking the standard deviation of GDP growth as an empirical analogue of σ, most countries have σ values well below 10%, making the risk term second order relative to GDP growth itself. For example, developed countries have GDP growth in the neighborhood of 2% and standard deviations of GDP growth around 2%. The growth term contributes 2% to the sustainability condition, while the risk term is 0.02%, or two orders of magnitude smaller in absolute value. Even if we assume a relatively high value of risk aversion, which makes it harder to meet the sustainability threshold, the growth term would still swamp the risk-adjusted volatility measure. As we will see, risk can indeed play a quantitatively important role if we introduce disaster risk to the model, but in the case of pure Brownian volatility, risk plays only a minor role in the general-equilibrium sustainability condition.[3]

[3] Economies can also achieve sustainability through lower consumption. In particular, if we set consumption to $\theta_{\text{con}} = \rho + g - \frac{1}{2}\sigma^2 < \rho$, then the risk-free rate is unaffected because consumption growth and volatility are the same and the rate of preference is the same.The inequality follows from the fact that $g < \frac{1}{2}\sigma^2$ when the constraint binds. This suggests a role for policies aimed at increasing national savings. However, the lesson from the endogenous growth literature is that policies ought to stimulate innovation and R&D investment, rather than capital accumulation.

2.3 Endogenous Growth and Investment

The main message of the general-equilibrium analysis is that sustainability is fundamentally a question of economic growth and risk. In a sense, this should not be surprising, given that these have always been among the driving forces in the literatures on development, growth, and finance. So far, however, we have modeled these as exogenous forces, effectively out of the hands of countries and international agreements. We now explicitly incorporate investment and growth.

There are a variety of ways in which one could endogenize growth in the model, including solving a competitive economy with AK production technology in the spirit of Romer (1986). Such a model would deliver the result that growth depends on a function of investment and capital depreciation. For the purposes of analyzing the impact of investment decisions on the sustainability constraint, we make the simplifying assumption that output is produced just using capital, K, which has a replacement cost of $\Phi(\iota)K$ (see Brunnermeier and Sannikov, 2014, 2016). In this setting, one can show that the growth rate is endogenous and given by

$$g(\iota) = \Phi(\iota) - \delta,$$

where ι is the investment rate and δ is the depreciation rate. In the presence of adjustment costs, $\Phi(.)$ would be both increasing and concave. Optimal investment satisfies Tobin's q:

$$Q\Phi'(\iota) = 1,$$

and consumption equals $K(1 - \iota)$, which is output net of investment. It is straightforward to show that the dividend yield in this setting equals ρ. Each unit of capital therefore generates dividends equal to ρQ, and this must equal the ratio of consumption to wealth:

$$\rho Q = 1 - \iota.$$

Excess returns are given by the sum of the dividend yield and growth less the risk-free rate:

$$\frac{1 - \iota}{Q} + g - r,$$

where the dividend yield reflects the investment rate.

We can solve in closed form by assuming the following functional form for the investment function. Let $\Phi(\iota) = \frac{1}{\phi}\log(1+\phi\iota)$. Then $\Phi' = 1/(1+\phi\iota)$, and optimal investment is given by

$$Q = 1 + \phi\iota \implies \iota = \frac{Q-1}{\phi}.$$

From market clearing, we have $\rho Q = 1 - \iota$, and therefore

$$Q = \frac{1+\phi}{1+\rho\phi}.$$

Hence, we have $\iota = (\rho - 1)/(1+\rho\phi)$, and the growth rate is given by

$$g = \frac{1}{\phi}\log\left(\frac{1+\phi}{1+\rho\phi}\right) - \delta.$$

While this is just one possible specification of endogenous growth, it illustrates a potential policy response regarding sustainability. If we consider a broader measure of investment that includes physical capital and human capital, we have a policy prescription that aligns with the fundamentals of economic growth from Solow to Romer. One of the most effective ways to ensure that expected welfare growth for future generations remains positive is to continue to invest in physical capital, education, and R&D. But that is just one component of the sustainability condition in general equilibrium. The other is the magnitude of risk, whether in the form of Brownian volatility, rare-disaster risk, or long-run regime switching, which we discuss in Section 3.

3 Disaster Risk, Long-run Risk, and "Green" vs. "Brown" Investments

The results so far have assumed that risk takes the form of Brownian volatility. This may understate the true nature of risk facing the world. Some of the greatest threats facing humanity are rare disasters—war, disease, and environmental catastrophes—which occur infrequently but with tremendous severity. In this section, we introduce two alternative specifications of risk: a jump process similar to the one in Campbell and Martin (2023) (i.e., *disaster risk*), and then a Markov switching process that allows for more permanent shocks to the economy, which we call *long-run risk*. As we will show, the type and the *stationarity* of risk matters in both partial and general equilibrium. We end this section with a discussion of green and brown investments, which differ in both growth and long-run risk. In contrast to the baseline specification, the introduction of green and brown investments provides a channel for the sustainability condition to affect portfolio choice, as well as consumption.

3.1 Disaster Risk

Jump risks may capture an important source of volatility, particularly with respect to events related to climate change. Campbell and Martin (2023) consider a flexible specification that includes both sources of risk. They find that jump risk does not change the fundamental message that risk allows the constrained consumption-wealth ratio to exceed the risk-free rate, while preserving the property that the optimal investment decision is unchanged by the sustainability constraint. In general equilibrium, however, we have seen that risk works in the opposite direction and actually makes it more difficult to satisfy the sustainability condition. Furthermore, we have seen that Brownian volatility is unlikely to be quantitatively important for most countries' sustainability constraints.

Here, we follow the general-equilibrium catastrophic-risk model in Pindyck and Wang (2013), who consider the question of how much societies should be willing to pay to avoid this particular type of risk. Catastrophic risk follows a Poisson process with mean arrival rate λ, with shocks destroying $1-X$ fraction of capital. To derive closed-form expressions, we suppose that X follows a Power distribution in $(0,1)$ with shape parameter $\beta > 0$, so that the density function (pdf) is $\zeta(X) = \beta X^{\beta-1}$, implying $\mathbb{E}[X] = \beta/(1+\beta)$.

By standard methodology, we can write the real interest rate r and the risk premium rp as

$$r = \rho + \gamma g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda \mathbb{E}\left[(X^{-\gamma}-1)\right],$$

and

$$rp = \gamma\sigma^2 + \lambda \mathbb{E}\left[(1-X)(X^{-\gamma}-1)\right].$$

As shown in Appendix B, the investor's optimal consumption-wealth ratio θ is

$$\theta = \rho + (\gamma-1)\left(g - \frac{\gamma}{2}\sigma^2\right) - \lambda\left(\mathbb{E}\left[X^{1-\gamma}-1\right]\right), \tag{13}$$

and the sustainability condition with $\alpha = 1$ is

$$r - \theta + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma}-1\right] \geq 0, \tag{14}$$

where $\hat{\mu} = rp + \lambda \mathbb{E}[(1 - X)]$. We can derive the general-equilibrium condition for sustainability by substituting the values of θ, r, $\hat{\mu}$, and rp from above into (14).

Proposition 2. *With CRRA utility with risk aversion γ and disaster risk, an economy satisfies the sustainability criterion in general equilibrium if*

$$g - \frac{1}{2}\gamma\sigma^2 - \lambda\frac{2\gamma - 1}{\beta - (\gamma - 1)} \geq 0. \tag{15}$$

As in Proposition 1, risk decreases the sustainability criterion in equilibrium, and for the same reason. While disaster risk increases the risk premium on the risky asset, it also decreases the risk-free rate. There is no returns channel (i.e., the positive effect due to the risk premium) because the expected return on wealth is still the expected growth rate of output. Meanwhile, the utility adjustment due to concavity is strictly negative and exacerbated by (larger) disaster risk. Thus, in general equilibrium, increasing risk—whether by increasing σ or by explicitly accounting for disaster risk—worsens sustainability.

While disaster and Brownian risk operate conceptually in the same way, explicitly incorporating disaster risk can have quantitative implications. We have already seen that the Brownian risk term has only a second-order impact on the sustainability constraint. The impact of disaster risk, however, depends on the value of risk aversion, the frequency of Poisson shocks, and the shape parameter governing the size of the shocks. As a starting point, we can take the calibration in Pindyck and Wang (2013), which sets $\gamma = 3$, $\lambda = 0.734$, and $\beta = 23$. With these parameters, the disaster risk term, $\lambda(2\gamma - 1)/(\beta - (\gamma - 1))$, is 17.5%, which will swamp the impact of growth and Brownian volatility in the sustainability constraint. This calibration, however, is geared toward equity risk and yields relatively frequent disasters, occurring every 1.4 years ($1/\lambda$).

If we instead consider disaster risks stemming from climate change, it might be more reasonable to imagine events that happen every 50 years, which corresponds to a $\lambda = 0.02$. Maintaining the assumption that $\beta = 23$, which yields approximately a 9% probability of losing at least 10% of output, we obtain a disaster risk term of about 0.5%, which would not be enough to cause the sustainability constraint to bind on an economy with a 2% growth rate and 2% volatility.

The sensitivity of these results to changes in the parameterization of disaster risks suggests that there may be considerable variation in the sustainability of different economies. For example, given moderate amounts of risk, a country with slower growth, such as the United States, could be in danger of reaching its sustainability constraint; a faster-growing country, such as China, might be safely above its sustainability constraint. These different experiences may help explain why some countries may be more willing than others to commit to international climate agreements. The slower-growth countries may recognize that they are at risk of leaving future generations with lower welfare, while higher-growth countries are almost certain to leave future generations with higher welfare.

The lesson here is that the calibration of disaster risk plays a crucial role in determining the sustainability of an economy—more frequent and more severe catastrophic risks make it less likely that a given economy is on a sustainable path. Our results also suggest that there could be a substantial return to mitigating or avoiding severe downside risks facing the climate and economy. One type of risk that is not captured in the models so far, however, is the path dependence of adverse outcomes. Climate models often have "tipping points," beyond which climates and economies suffer irreversible changes. It is worth considering how these kinds of risks affect the sustainability criteria in both partial and general equilibrium.

3.2 Long-run Risk

Disaster risk admits the possibility of large, discontinuous shocks within a stationary framework. These shocks can be severe, but they are also transitory. Scientists, however, have documented a growing list of potentially irreversible "tipping points" that could both amplify current threats of climate change and have lasting effects on the planet (Lenton et al., 2008). A formal model of climate tipping points is beyond the scope of this paper, but we can examine how nonstationary, long-run risks affect the likelihood that a given consumption path is sustainable.

Pritchett (2003) and Jerzmanowski (2006) have stressed that economic performance is not just a story about average growth rates over time, but about different growth regimes that influence growth rates over long periods of time. The possibility of regime change introduces the possibility of poverty traps, stagnation, and prolonged expansions. In the context of sustainability, similar dynamics may apply to tipping points in the environment or breakthroughs in technology that open up lasting opportunities for consistent growth. In order to examine the impact of these sustained risks, we introduce a Markov switching model in the spirit of Chari et al. (1996). As in their model, we can think about an economy that switches between a "good regime," in which distortions are low, and a "bad regime," in which distortions are severe.

Since the analysis of this kind of risk is new to the literature on sustainability constraints, we first examine the partial-equilibrium case and then show how things change in general equilibrium.

We find that long-run risk *tightens* the sustainability constraint in partial equilibrium—in contrast to stationary risk, which relaxes the sustainability constraint. For tractability, we consider only Brownian risk (no disaster risk) and log utility.

3.2.1 Partial Equilibrium

The Markov switching process has two states, 0 and 1, which affect the excess return on the risky asset. In particular, we assume that state 1 is the good state, with $\mu_1 > \mu_0$, with regime switching governed by a Poisson process with rate λ.[4]

We first solve the standard optimization problem. Let the value function in state i be V_i. Let α_i denote the portfolio and let c_i denote consumption. The investor HJB is

$$\rho V_i = \max_{c_i,\alpha_i}\left\{\log(c) + V_i'(r + \alpha\mu_i - c_i/W)W + \alpha_i^2\frac{\sigma^2}{2}W^2V_i'' + \lambda\left(V_j - V_i\right)\right\}.$$

Optimizing over c_i and α_i, we obtain the familiar conditions: $c_i = \rho W$ and $\alpha_i = \mu_i/\sigma^2$. The presence of Markov shocks does not change the decisions about consumption and portfolio choice. It does, however, affect welfare, which is given by:

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR_i^2}{2}\right) - \frac{\lambda}{2\rho^2(\rho+2\lambda)}\left(SR_i^2 - SR_j^2\right). \tag{16}$$

In order to see the impact of the Markov switching, we can define

$$\overline{SR_i^2} \equiv \frac{(\rho+\lambda)SR_i^2 + \lambda S_j^2}{\rho + 2\lambda},$$

which is a weighted average of the two Sharpe ratio terms, with higher discount rates putting greater weight on the risk associated with the current Markov state. Substituting into equation (16), we have

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{\overline{SR_i^2}}{2}\right), \tag{17}$$

which looks like equation (2), except that the value function depends on the state, and the Sharpe ratio is the weighted average of the ratios in the two states.

Before we introduce the sustainability constraint, we characterize welfare for given values of θ_i and α_i. As before, we define $\nu \equiv r + \alpha\mu - \theta - \alpha^2\sigma^2/2$. For a given pair $\{\theta_i, \alpha_i\}$, welfare is given by

[4]With some abuse of notation, we use λ to denote the Poisson arrival rate for either disaster risks or long-run risks.

$V_i = (\log W)/\rho + b_i$, with

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{\lambda}{\rho + 2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i - \nu_j)\right). \tag{18}$$

The expected change in welfare is the drift in wealth, minus the Itô term reflecting risk, plus the Poisson term:

$$\begin{aligned}\mathbb{E}[\mathrm{d}V_i] &= \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i - b_j), \\ &= \frac{\nu_i}{\rho} - \lambda(b_i - b_j),\end{aligned} \tag{19}$$

where

$$(b_i - b_j) = \frac{1}{\rho + 2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i - \nu_j)\right). \tag{20}$$

Optimality requires $\theta_i = \theta_j = \rho$ and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. By equation (20), we have

$$\mathbb{E}[\mathrm{d}V_i] \propto r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho + 2\lambda)}\left(SR_i^2 - SR_j^2\right).$$

Sustainability requires that the expected drift in welfare is non-negative, which yields the following:

Proposition 3. *In partial equilibrium with long-run risk, the sustainability constraint will bind whenever*

$$r + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho + 2\lambda)}\left(SR_i^2 - SR_j^2\right) < \rho. \tag{21}$$

Whereas Brownian and disaster risk provide more space for sustainability in partial equilibrium, long-run risk pushes in the opposite direction. The possibility of moving to a state with lower returns (lower Sharpe) makes it more likely that the sustainability constraint binds. The Sharpe ratio continues to provide additional space in the sustainability constraint, but the probability of moving to a lower state, captured by λ, makes sustainability less likely, and this risk is not reflected in the risk premium.

Proposition 3 highlights the importance of understanding precisely what types of risks face the global economy and how they affect sustainability concerns. Stationary risks, even if they concern rare disasters, are reflected in risk premia and therefore provide additional space for sustainability through the returns channel, leading to higher expected wealth growth. In contrast, nonstationary risks such as long-run risks, which may better capture the types of risks inherent in climate change, are not reflected in risk premia and therefore do not contribute to higher expected returns on wealth. Even in partial equilibrium, long-run risks operate entirely through a welfare adjustment that tightens sustainability considerations.

3.2.2 General Equilibrium

The natural way to incorporate Markov switching into the general-equilibrium framework is by supposing that growth rates change with the state, letting $g_1 > g_0$. This has the effect of also allowing r_i to change with the state. Because fundamental risk σ determines excess returns, we will have $SR_1 = SR_2 = \sigma^2$ in both states. (Note the contrast with partial equilibrium.) With changing growth rates, we have

$$r_i = \rho + g_i - \sigma^2 \implies \nu_i = g_i - \sigma^2/2.$$

We can easily characterize when the economy is on a sustainable path in state 1:

Proposition 4. *In general equilibrium with long-run risk, the sustainability constraint will bind in state* 1 *if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho + 2\lambda}(g_1 - g_0) < 0. \tag{22}$$

Intuitively, the prospect of switching to a low-growth regime makes it more likely that an economy will run up against its sustainability constraint. However, as we discussed in the context of the rare-disaster risk, it's plausible that λ is not very high—perhaps on the order of 2% per year. In that case, the growth loss $g_1 - g_0$ has to be substantial for the Markov risk alone to create a binding sustainability constraint.

3.2.3 Downside Risk

For tractability, we have so far assumed that Markov risk is symmetric. In reality, we are likely concerned about the prospect of asymmetric *downside* risk: the possibility of moving permanently into a low-growth regime. Now suppose that with probability λ the economy moves from the high-growth state 1 to the low-growth state 0 and then stays there forever. The results of the previous analysis go through with a minor change to the denominator of the Markov term to reflect that state 0 is an absorbing state.

For expositional purposes, suppose initially that the sustainability constraint does not bind in state 0. As shown in Appendix C, we can solve for the following modified sustainability condition:

Proposition 5. *Suppose the sustainability constraint does not bind in* 0*. In general equilibrium, the sustainability constraint will bind in state* 1 *if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho + \lambda}(g_1 - g_0) < 0. \tag{23}$$

This is nearly identical to the condition in Proposition 4, except that there is a larger coefficient on the growth loss term, $g_1 - g_0$, reflecting the consequences of moving to a permanently lower growth regime.

What is the quantitative significance of long-run downside risk? As a rough calibration, we can set $g_1 = 2\%$, $\sigma = 2\%$, and $\rho = 5\%$. Let $\Delta \equiv g_1 - g_0$ be the growth at risk in switching from g_1 to g_0. Then for the sustainability constraint to bind, we need

$$0.02 - \frac{1}{2}(0.02)^2 < \frac{\lambda}{0.05 + \lambda}(\Delta).$$

Even if we assume a relatively high value of $\lambda = 10\%$, we need a Δ of almost 3% in order for the sustainability constraint to bind. Given our assumption of 2% growth in the good state, this would amount to a *negative* 1% growth rate in the bad state. If λ is closer to 2%, as we argued before, the difference in growth rates would have to be over twice as large. Regardless of the precise calibration, however, it is clear that long-run risk makes it more likely that the sustainability constraint binds.

The analysis so far has assumed that the sustainability constraint does not bind in state 0. But for reasonable calibrations of the model's parameters, we have seen that $g_0 \leq 0$, in which case the sustainability constraint *will always bind* in state 0. The optimal consumption in state 0 is then adjusted to

$$\theta_0 = \rho + g_0 - \frac{\sigma^2}{2} < \rho,$$

which means that $\log(\theta_0)$ is lower than $\log(\rho)$, implying an additional welfare loss when the Markov shock occurs. Importantly, the failure to meet the sustainability constraint after downside risk realizes increases the urgency to pursue sustainability in the present. This dynamic makes the sustainability constraint in state 1 even more likely to bind. It immediately follows that if the sustainability constraint does bind in state 0, we have to update the sustainability constraint in state 1 as follows:

Proposition 6. *Suppose* $g_0 \leq \sigma^2/2$*, so that the sustainability constraint binds in state* 0*. In general* *equilibrium, the sustainability constraint will bind in state* 1 *if*

$$g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+\lambda}(g_1 - g_0) - \frac{\rho\lambda}{\rho+\lambda}\left(\log\rho - \log\left(\rho + g_0 - \frac{\sigma^2}{2}\right)\right) < 0. \tag{24}$$

The additional term that comes from distorted consumption in 0 is not very large, so quantitatively this is not a very significant additional term. Nevertheless, it is important to keep in mind that the possibility of a binding constraint in the bad state of the world has dynamic implications for the likelihood of a binding constraint in the good state. In addition, our example has assumed Brownian risk alone, which biases against a binding constraint. If we instead assumed a greater role for stationary risk, either by increasing σ or reintroducing jump risk, the sustainability condition in equation (23) would bind with even smaller values of Δ. For example, if we set $\sigma = 10\%$ in our previous example, with $\lambda = 10\%$, then equation (23) would bind whenever $\Delta > 2.25\%$. If current growth is $\approx 2\%$, this would imply negative growth in the downside-risk state, in which case the sustainability constraint would bind going forward.

In contrast to the case of stationary risk (whether Brownian or disaster volatility), long-run risk makes sustainability less likely in both the partial- and the general-equilibrium settings. It strengthens the importance of risk in the general-equilibrium world, and it decreases the gap between the consumption-wealth ratio and the risk-free rate in the partial-equilibrium framework.

3.3 Green and Brown Technologies

An important consideration for sustainability is how different technologies and investment options affect long-run outcomes. The model in Section 2.3 assumed that there was only one type of investment, which affected the sustainability condition through its impact on overall economic growth. In many settings, however, the type of investment may be as important as the amount of investment when it comes to sustainability. In the context of climate change, for example, Acemoglu et al. (2016) have argued that it is essential to distinguish between clean technologies and dirty technologies in modeling investments in the energy sector. This argument is not confined to climate change. Almost all investments come with some kind of externality, whether positive or negative. Some of the most salient examples include human capital investment, infrastructure spending, R&D, military spending, disaster preparation, and institutions aimed at risk mitigation or sharing.

We consider a simple version of the model with two linear technologies (brown and green).

Brown technologies offer high returns today at the expense of long-run risk, externalities, or resource depletion. *Green* technologies, in contrast, offer lower expected returns but do not create negative externalities or additional risks.[5] To simplify the analysis and clearly illustrate how sustainability concerns could distort the optimal investment decision, we make the following assumptions so that in equilibrium investors will invest in a single technology. Specifically, the two linear technologies produce perfectly correlated dividends with identical Brownian volatilities, σ. Assuming perfect correlation is a simplification that implies that there is no diversification motive for investors, allowing us to focus on the single preferred investment for an investor. We assume that investors choose to operate the technologies (i.e., to "plant the trees") so in equilibrium the supply of each tree will be endogenous and potentially zero.

The two technologies differ in terms of their growth rates and long-run risks. The brown technology initially offers a higher expected dividend growth rate, g_1^B, while the green technology comes with a lower expected dividend growth, $g_1^G < g_1^B$. However, the brown technology comes with the risk of a disastrous long-run event, which we model as a Poisson process with arrival rate λ that decreases growth to $g_0 < g_1^G$ going forward. The green technology has no long-run risk (i.e., $\lambda = 0$ for this tree). We suppose $g_1^G > \frac{1}{2}\sigma^2$ so that the green technology is sustainable. We discuss using long-run risk to model negative externalities in greater detail later, after characterizing the equilibrium.

In this simple economy, it is easy to confirm that in equilibrium the tree produced by each technology would sell for the same price per dividend, $Q^i = 1/\rho$ for $i = G, B$, which is determined by the market clearing for consumption and is not altered by the presence of long-run risk. An immediate implication is that the asset prices grow at the dividend growth rates, so that the expected return for each tree i is given by $\rho + g_1^i$. Thus, the brown technology offers a higher expected return for the same Brownian risk σ. Thus, given any risk-free rate r, investors would strictly prefer to invest entirely in the brown technology.

This result can change, however, in the presence of a sustainability condition. To see this, note that sustainability in this setting requires that:

$$g_1^B - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+\lambda}\left(g_1^B - g_0\right) - \frac{\rho\lambda}{\rho+\lambda}\left(\log\rho - \log\left(\rho + g_0 - \frac{\sigma^2}{2}\right)\right) \geq 0. \tag{25}$$

[5] While the names of these technologies strongly suggest an application to the problem of climate change, we emphasize that they are meant to capture any tradeoff between technologies that offer higher returns with negative externalities and technologies that offer lower returns with either zero or positive externalities.

If the disaster risk is low, the brown investment would satisfy the sustainability condition. However, if the long-run risk is sufficiently severe, sustainability considerations would tilt the optimal investment toward the green technology. For example, if λ is sufficiently high and g_0 is sufficiently low, the economy will end up violating the condition given by inequality (25). In this case, sustainability would require a shift away from the brown technology and toward the green one. In this setting, the sustainability constraint changes the optimal portfolio decision, a result that does not emerge in a model with only one risky investment (Campbell and Martin, 2023).

In the real world, we do not observe the kind of all-or-nothing allocations between green and brown investments. There are naturally a large number of such investments, which fall along a spectrum of environmental damage and dividend growth. One can imagine investors choosing from a menu of investments with differing degrees of green and brown characteristics. For example, suppose investments i offer a pair of (λ^i, g_1^i). Then the socially optimal investment would depend on the size of the long-run risk and the relative dividend growth rates of the investments. Choosing from a menu of investment options would likely result in an interior socially optimal investment that exactly satisfies the sustainability condition.

The result that sustainability concerns change the investment decision would also apply to decisions regarding abatement. Suppose that investors can choose to make costly abatement decisions in order to make a brown investment greener. We could model such decisions as costly efforts to decrease the long-run risk λ at the cost of decreasing growth g. Since safe and risky returns depend on contemporaneous growth and risk, in a competitive equilibrium agents would find no reason to make the costly investment to decrease long-run risk. However, the socially optimal decision would pursue abatements until the sustainability condition is satisfied.

Brown and green technologies would also coexist if we instead supposed that the two technologies represented existing trees, or capital, that must be held in equilibrium, as in Section 2.3. The result in this case is that the green tree would trade at a discount to the brown tree and the economy would invest more toward brown capital accumulation. To simplify, assume away capital accumulation for the moment. Market clearing for assets would require that expected returns be equal, $1/Q^B + g_1^B = 1/Q^G + g_1^G$. These imply that $Q^B > Q^G$ since $g_1^G < g_1^B$. Adding endogenous capital accumulation via investment, as in Section 2.3, would not change this result. We would instead have $(1-\iota^B)/Q^B + g_1^B = (1-\iota^G)/Q^G + g_1^G$, and so in equilibrium it must be that $Q^B > Q^G$. (Solving exactly would depend on the investment technology Φ.) The economy would invest more capital in the brown technology ($\iota^B > \iota^G$) because of the higher price. Asymptotically, all output in the economy would be produced by the brown technology.

In addition, we have only focused on a general economic risk associated with the brown asset. We have modeled the brown technology as having an investment-specific long-run risk. Since our economy features a single investment in equilibrium, this is without loss of generality. However, the negative externalities from brown technologies should likely be modeled as creating some long-run risks for all technologies in the economy, both green and brown. In reality, the type of risk is likely to matter. Karydas and Xepapadeas (2019), for example, distinguish between the *physical* risks associated with climate change and the *transition* risks associated with stringent policy reforms. In that setting, the physical risks affect both assets, while the transition risks apply just to the brown assets.

Aggregate risks such as physical climate risks would decrease productivity not only for brown investments but for green ones as well, and the level of long-run risk would likely depend on the aggregate level of brown investment. This would underscore the *externality* feature of the technology. Even in our simple setting, in which the degree of long-run risk in the brown technology does not create externalities for the green technology or depend on the aggregate level of investment, sustainability concerns still distort the investment decision made by the competitive equilibrium. Adding in more features of the externality in a model with multiple investments would provide additional reasons for sustainability concerns to affect the socially optimal investment decision.

Finally, we have also abstracted away from the potential for diversification to play a role. Hambel et al. (2021), however, show that diversification can add an important layer of nuance to the investment decision. We could easily modify our preceding example as follows. Suppose the initial growth rates are the same, $g_1^B = g_1^G$, and the dividend processes are independent. Then the privately optimal investment would allocate capital equally to both technologies due to diversification motives. However, if the brown technology has long-run risk, then the socially optimal investment would put less than half of the capital in the brown technology. If long-run risk is not so severe, then the optimal allocation would remain interior. While environmental concerns would push the economy toward a sharp reduction in brown technologies, diversification would still lead to a balance between both asset types.[6]

[6]If we depart from the general-equilibrium setting, it is possible that climate change risks can indeed lead to reductions in asset prices, with associated increases in risk premia (Bansal et al., 2016).

4 Financial Depth and Integration

This section considers growth and financial development in more detail and shows how they can be incorporated directly into a general-equilibrium sustainability framework. So far, our models have assumed frictionless financial markets. In reality, countries have different levels of financial development, with some countries facing significant gaps in the return on savings and the cost of borrowed funds and others creating a wide span of assets available for domestic and global savings. Given the importance of financial risk in the general-equilibrium sustainability model, it is worth exploring the roles of financial depth and financial integration.

4.1 Financial Depth

There are a variety of ways to incorporate depth into the sustainability model. One way is to assume that countries differ in their ability to supply financial assets to savers, which has implications for the stability of consumption flows. Following Caballero, Farhi, and Gourinchas (2008), suppose that a country has "financial depth" $\kappa \in (0, 1)$, which captures the fraction of future cashflows that can be capitalized as assets. *Financial depth*, or stability, thus measures the suitable supply of assets available for savings.

Total output is given by D_t, a fraction κ of which comes from a tree and the remainder of which comes from endowments to new agents, which are saved until death. Agents are born and die at the same rate, ρ, and thus, existing agents consume a fraction ρ of their wealth. Critically, existing wealth does not include future endowments (i.e., the wealth of agents yet to enter). We modify equilibrium as follows. The return on the tree satisfies

$$\frac{\kappa}{Q} + g - r = \sigma^2,$$

where the κ numerator reflects the smaller dividend. Market clearing for assets requires that all existing wealth is held in the tree—hence, $QD = W$. Market clearing for consumption is $\rho W = D$, which implies that $\rho Q = 1$. This means we have

$$\rho\kappa + g - r = \sigma^2 \implies r = \rho\kappa + g - \sigma^2 < \rho + g - \sigma^2.$$

As in Caballero, Farhi, and Gourinchas (2008), we have the result that lower financial depth reduces the risk-free rate. This has an immediate implication for the sustainability of consumption decisions: a closed economy with a limited ability to supply risk-sharing assets will be more likely to face a binding sustainability constraint. Recall that sustainability in general equilibrium requires that:

$$r - \rho + \sigma^2 - \frac{1}{2}\sigma^2 \geq 0.$$

Substituting the modified value of the risk-free rate, we have

$$g - \frac{1}{2}\sigma^2 - \rho(1-\kappa) \geq 0,$$

requiring

$$g \geq \frac{1}{2}\sigma^2 + \rho(1-\kappa).$$

Our previous general-equilibrium condition is equivalent to assuming that $\kappa = 1$. Economies with limited abilities to generate risk-sharing assets will therefore face more binding sustainability constraints. However, this also provides a potential policy vehicle for improving sustainability. In addition to increasing growth through investments in physical and human capital, countries can relax their sustainability constraints through increasing the depth of financial markets.

4.2 Financial Integration

So far, we have considered a closed-economy perspective. The impact of financial depth, however, likely depends on the level of financial integration with the rest of the world. A simple way to see this point is to consider two countries that differ in both growth and financial depth, g and κ. As we have seen, the autarkic interest rates would be

$$r_1 = \rho\kappa_1 + g_1 - \sigma^2, \quad r_2 = \rho\kappa_2 + g_2 - \sigma^2.$$

As an important illustrative example, we could let 1 denote the U.S. and 2 denote China. Suppose that $r_1 > r_2$ with $g_2 \geq g_1$, which reflects the fact that China's growth rate has been well above that in the U.S., while its financial depth has been notably lower. The result, not surprisingly, is a flow of capital from China to the U.S., which we have seen in the data. As a simplification, assume that China and the U.S. are the same size and make up the entire world. Then we have a world interest rate

$$r = \rho\bar{\kappa} + \bar{g} - \sigma^2,$$

where bars denote averages. In this case, the sustainability constraint requires

$$\bar{g} \geq \frac{1}{2}\sigma^2 + \rho(1 - \bar{\kappa}).$$

Financial integration mitigates the impact of limited financial depth on China's sustainability constraint, while it reduces the benefit of higher financial depth in the U.S..

5 Policies to Address Sustainability

Regardless of the specification, the general-equilibrium framework provides a clear message about policies aimed to improve sustainability: countries can take measures to improve growth or to reduce the volatility of output through mitigating catastrophic risk, reducing financial frictions, or developing financial markets.

5.1 Mitigating Catastrophic Risk

We have already seen that catastrophic risk has the potential to overwhelm the importance of growth and Brownian risk in the sustainability condition. While the model assumes an exogenous process for disaster risk, in terms of both the frequency and the severity of events, the process can be influenced by policies aimed at lowering the likelihood of catastrophic outcomes and mitigating the consequences of events that do happen. In the language of the model, we are looking for policies that lower the arrival rate, λ, of events, policies that alter the power distribution parameter, β, and policies that change the fraction of capital destroyed for a given shock, X.

While there is no shortage of catastrophic risk facing the world, perhaps the two most salient examples are climate change and nuclear weapons. In both of these cases, countries have a limited ability to control the parameters governing the stochastic process alone but more scope to develop policies aimed at mitigating the impact of shocks, particularly in the case of climate change. Thus, we should expect to see a combined emphasis on treaty-based solutions, on the one hand, and investments focused on mitigation, on the other.

The Netherlands poses an extreme example in the case of climate change. The country's low-lying terrain and abundance of waterways exposes the Dutch to extreme and rising risks associated with flooding. The Netherlands has invested in a combination of infrastructure and management strategies to mitigate the impact of flood risk, including dike renovation, floating homes, and modernized agriculture. Also, in 2019, the Netherlands passed the Climate Act, which commits the country to achieving a 95% reduction in greenhouse gasses by 2050. The mitigation strategies directly address the impact of the highest-risk climate shocks, while the climate agreement aligns Dutch policy with coordinated targets in Europe.

The Climate Act underscores a missing element in our general-equilibrium model of sustainability: the role of externalities and the need for coordinated responses. Most catastrophic risks have causes and consequences that both cross international borders. The standard approach to modeling externalities at the global level has been the use of Integrated Assessment Models (IAMs), which combine a model of economic growth with an environmental externality that can be influenced by decisions about production and technology.[7] While such a model is beyond the scope of our sustainability framework, it is worth noting that the presence of externalities would almost surely change some of the model's conclusions. For example, in both Campbell and Martin (2023) and our own framework, the sustainability constraint does not change the optimal allocation between the safe and the risky asset. This may not be true in the presence of externalities, which might make the risky investment option less attractive in the presence of a constraint. Furthermore, the presence of externalities also places a premium on coordinated solutions to climate change, including William Nordhaus's "Climate Club" proposal (Nordhaus, 2015), international climate agreements (Harstad, 2016), and technology-sharing and trade policies (Weber and Peters, 2009).

5.2 Technology, Investment, and Human Capital

In a sense, it is inevitable that a model of sustainable growth would bring us back to the seminal contributions of Solow (1956), Romer (1990), and Mankiw et al. (1992), which place technology, investment, and human capital at the center of economic growth. Along with direct measures to avoid or mitigate the risks associated with climate change, these factors will continue to play a fundamental role in determining whether expected welfare for future generations increases or decreases—both at the country level and for the world as a whole. In addition, these same factors are essential for building the capacity to directly address challenges posed by climate change, demographic change, and persistent warfare.

These factors can also play a critical role in reducing the probability and severity of catastrophic events. Transitioning to clean energy, reducing the likelihood of another Fukushima, or securing stockpiles of nuclear weapons will all require strategic investments in technology, infrastructure, and education. While our model does not allow for explicit decisions about investing in, for ex-

[7] See Nordhaus (2017) for a discussion and comparison of various IAMs.

ample, clean vs. dirty technologies, it is easy to imagine how directed technical change (Acemoglu et al., 2012, 2016) might provide an additional lever to reach or maintain a sustainable trajectory. Similarly, human capital can lead to both an increase in long-run growth (Barro, 2001; Jones and Romer, 2010), which relaxes the sustainability constraint, and a reduction in energy consumption (Shahbaz et al., 2022), which has a direct effect on the sustainability of current decisions about spending and asset allocation.

5.3 Financial Development and Integration

The policies we have reviewed focus on reducing catastrophic risk and increasing the rate of economic growth, both of which tend to move an economy further above its sustainability constraint. Financial markets offer up the possibility of improving the management of risks, increasing the set of productive investments, and raising the return on wealth. The extensions in Section 4 showed some specific ways in which one might introduce financial depth and integration into the sustainability framework, but there are other possibilities.

The notion that we need more financial innovation is often met with skepticism, particularly after the adventures in securitization that preceded the global financial crisis of 2008. There is little doubt, however, that financial development is crucial for economic growth (Levine, 1997). As Levine (1997) notes, "Theory suggests that financial instruments, markets, and institutions arise to mitigate the effects of information and transaction costs. Furthermore, a growing literature shows that differences in how well financial systems reduce information and transaction costs influence saving rates, investment decisions, technological innovation, and long-run growth rates..." In the language of our framework, financial development can increase growth and reduce the severity of risk. While it is challenging to disentangle the causal relationship between financial development and these outcomes, it seems clear that higher levels of financial development can help countries attain or remain on a sustainable path of growth.

As we have seen in Section 4, the lessons for financial integration are less clear. In the model setting, the impact of integration depends on the level of economic development of the respective countries. In practice, there is mixed evidence on the tendency for financial integration to reduce volatility. Kose et al. (2003), for example, find that the volatility of consumption increased relative to the volatility of income for more financially integrated countries during the 1990s, a period of increased integration. Also, the extent to which financial integration contributes to positive growth and reduced volatility may be due more to indirect channels, such as the impact of openness on institutions and governance, than to the direct effects on saving and diversification (Kose et al., 2009).

We have chosen to highlight financial approaches to reducing volatility, but the potential policy space is much richer than that. The connection between economic development and volatility is complex and involves the sectoral composition of production (Koren and Tenreyro, 2007), institutions (Acemoglu et al., 2003), and trade openness (Giovanni and Levchenko, 2009), among other factors. Any policies or institutions that reduce the volatility of consumption will help push countries further above their sustainability constraints.

6 Conclusion

What does it mean for an economy to be "sustainable" in general equilibrium? Defining sustainability as a nondecreasing path of expected future welfare, this paper argues that sustainability fundamentally depends on the forces of economic growth and volatility. While this may not seem surprising, in light of the central role these factors have played in macroeconomics, development, trade, and finance, it is not the result that one obtains from a partial-equilibrium framework. There, as Campbell and Martin (2023) have shown, risk allows the sustainable ratio of consumption to wealth to be higher than it would be in a world without risk. Effectively, the positive effect of the risk premium on future wealth more than offsets the negative effect of volatility on welfare due to risk aversion. An important exception to this occurs in the case of nonstationary risk, such as long-run risks that could manifest in persistently or permanently lower growth rates. In both partial- and general-equilibrium settings, nonstationary risks make it more likely that a sustainability constraint binds. Our results highlight the importance of understanding precisely the nature of risks facing the global economy.

In general equilibrium, expected returns are determined by the economic growth rate, and the risk-free rate adjusts to absorb the risk premium. Here, stationary risk only affects sustainability through its impact on the concavity of utility (which decreases the sustainable consumption-wealth ratio), and the only way to increase the expected return on savings is through increasing economic growth. Nonstationary risk makes the downside state of the world more likely, which further reduces expected welfare. In a sense, general equilibrium returns the conversation to the same factors that have been fundamental to growth and development all along: investment, education, technology, and financial depth and integration.

This, then, raises the question of which framework most accurately captures the tradeoffs between growth and sustainability. The answer likely depends on the degree of aggregation: whether we are considering the world as a whole, an individual country, or an SOE or institution within a country. The smaller the unit of aggregation, the more reasonable it is that the presence of a risky investment opportunity relaxes a sustainability constraint. This would be true for individuals, institutional endowments, pension funds, SOEs, and perhaps even for some sovereign wealth funds. One of the main lessons of Campbell and Martin (2023) is that, for these agents, sustainability considerations do not distort the allocation of capital (between safe and risky investments), and so, subject to standard externalities, market returns can and should serve as signals for how to invest capital. Nonetheless, even for these countries, long-run risks would pose a challenge for sustainability. At the highest levels of aggregation, however, general equilibrium may be a more appropriate framework. The world economy as a whole cannot escape the fundamental forces of growth and volatility.

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Appendices

A Derivations and Proofs for Section 2

Partial Equilibrium We first solve the standard optimization problem. The HJB is

$$\rho V = \max_{c,\alpha} \left\{ \log(c) + V'(r + \alpha\mu - c/W)W + \alpha^2 \frac{\sigma^2}{2} W^2 V'' \right\}.$$

With log utility, the value function can be written as $V = \frac{\log W}{\rho} + b$, so that $V' = \frac{1}{\rho W}$ and $V'' = -\frac{1}{\rho W^2}$. Plugging in, we have

$$\rho V = \max_{c,\alpha} \left\{ \log(c) + \frac{1}{\rho}(r + \alpha\mu - c/W) - \alpha^2 \frac{\sigma^2}{2\rho} \right\},$$

and taking first-order conditions, we have the standard conditions

$$\frac{1}{c} = \frac{1}{W\rho} \implies c = \rho W, \quad \frac{1}{\rho}\mu = \alpha \frac{\sigma^2}{\rho} \implies \alpha = \frac{\mu}{\sigma^2}. \tag{26}$$

The value function therefore satisfies

$$\log W + \rho b = \log(\rho) + \log W + \frac{1}{\rho}(r + SR^2 - \rho) - \frac{SR^2}{2\rho},$$
$$\rho b = \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR^2}{2}\right).$$

Hence, we have

$$V = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR^2}{2}\right).$$

When the sustainability constraint binds, we then have the following condition on θ, α:

$$r + \alpha\mu - \theta - \frac{1}{2}\alpha^2\sigma^2 = 0 \implies \nu = 0. \tag{27}$$

The sustainability constraint imposes $\nu = 0$. We can plug into equation (35) to write b as

$$\rho b = \log(\theta) + \frac{1}{\rho}\nu = \log(\theta).$$

Maximizing welfare comes down to maximizing θ, subject to the sustainability constraint. This is why the optimal portfolio decision is the same as before. The constraint does not distort the risk-reward tradeoff from investment, so the optimal allocation provides the highest possible consumption rate. We show this now. For a given consumption and portfolio choice θ, α, welfare is given by

$$V = \frac{\log W}{\rho} + \frac{\log(\theta)}{\rho} + \frac{1}{\rho^2}\left(r - \theta + \alpha\mu - \frac{1}{2}\alpha^2\sigma^2\right). \tag{28}$$

We can write the HJB:

$$\rho A \log W + \rho b = \log(\theta) + \log W + A(r + \alpha\mu - \theta) - \alpha^2\frac{\sigma^2}{2}A.$$

We immediately have $\rho A = 1$, as before. Continuing

$$\rho b = \log(\theta) + \frac{1}{\rho}\left(r + \alpha\mu - \theta - \alpha^2\frac{\sigma^2}{2}\right).$$

Note that maximizing welfare at this stage corresponds to choosing θ, α to maximize b. Doing so, we are guaranteed to get the same FOC as before, $\theta = \rho$ and $\alpha = \frac{\mu}{\sigma^2}$. Again, it becomes clear that maximizing welfare with a binding constraint comes down to maximizing θ subject to the sustainability constraint. This is why the optimal portfolio decision is the same as before. The constraint does not distort the risk-reward tradeoff from investment, so the optimal allocation provides the highest possible consumption rate.

CRRA utility with $\gamma > 1$ With CRRA utility, there are a few changes. First, the welfare function is now $V = AW^{1-\gamma}$. The optimality conditions are now

$$\alpha = \frac{\mu}{\gamma\sigma^2}$$

and

$$\begin{aligned}\frac{c}{W} &= \frac{\rho + (\gamma - 1)(r + \alpha\mu - \frac{1}{2}\gamma\alpha^2\sigma^2)}{\gamma}, \\ &= \frac{\rho + (\gamma - 1)(r + \frac{1}{2\gamma}SR^2)}{\gamma}.\end{aligned}$$

Given the welfare function, the natural consideration for sustainability is $\mathbb{E}[\mathrm{d}V/V]$. By Itô's Lemma, the sustainability constraint is modified to reflect risk aversion (Campbell and Martin, 2023):

$$\mathbb{E}\left[\mathrm{d}V/V\right] \propto r - \theta + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2. \tag{29}$$

Note that the portfolio terms $r + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2$ are in the numerator of consumption, and hence they can be written as $r + \frac{1}{2\gamma}SR^2 - \theta$. This also means that when we plug in for consumption, the terms will combine nicely.

Proposition 7. *With CRRA utility, the sustainability constraint is given by*

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0. \tag{30}$$

Proof. At the optimal, the sustainability constraint can be written as

$$\begin{aligned}
\mathbb{E}[\mathrm{d}V/V] &\propto r + \frac{1}{2\gamma}SR^2 - \frac{\rho + (\gamma-1)(r + \frac{1}{2\gamma}SR^2)}{\gamma}, \\
&= \frac{\gamma}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right) - \frac{\rho}{\gamma} - \frac{\gamma-1}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right), \\
&= \frac{1}{\gamma}\left(r + \frac{1}{2\gamma}SR^2\right) - \frac{\rho}{\gamma}, \\
&= \frac{1}{\gamma}\left(r - \rho + \frac{SR^2}{2\gamma}\right).
\end{aligned}$$

Since $\gamma > 0$, this yields the result. □

Recursive Preferences We also solve with recursive Epstein-Zin (EZ) preferences with EIS ϵ.

Proposition 8. *With recursive (Epstein-Zin) utility, optimal consumption is given by*

$$c/W = \epsilon\rho + (1-\epsilon)\left(r + \frac{1}{2}\gamma\alpha^2\sigma^2\right) = \epsilon\rho + (1-\epsilon)\left(r + \frac{1}{2\gamma}SR^2\right)$$

and the sustainability constraint is given by

$$r - \rho + \frac{SR^2}{2\gamma} \geq 0. \tag{31}$$

Proof. Consumption is standard and follows from plugging in $\alpha = \mu/(\gamma\sigma^2)$. At the optimal, the sustainability constraint can be written as

$$
\begin{aligned}
\mathbb{E}[\mathrm{d}V/V] &\propto r + \alpha\mu - \gamma\frac{1}{2}\alpha^2\sigma^2 - \left(\epsilon\rho + (1-\epsilon)\,(r + \frac{1}{2}\gamma\alpha^2\sigma^2)\right), \\
&= r + \frac{1}{2}\gamma\alpha^2\sigma^2 - \left(\epsilon\rho + (1-\epsilon)\,(r + \frac{1}{2}\gamma\alpha^2\sigma^2)\right), \\
&= \epsilon\left(r + \frac{1}{2}\gamma\alpha^2\sigma^2\right) - \epsilon\rho, \\
&= \epsilon\left(r - \rho + \frac{SR^2}{2\gamma}\right),
\end{aligned}
$$

and the result follows since $\epsilon > 0$. □

Hence, the condition for a binding sustainability constraint is the same with CRRA or EZ preferences.

General Equilibrium with Recursive Preferences With recursive preferences, the risk-free rate satisfies

$$
r = \rho + \frac{1}{\epsilon}\mu_c - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma_c^2,
$$

where μ_c and σ_c are the expected growth rate and volatility of consumption. The excess risk-premium equals $\gamma\sigma^2$, which means $SR = \gamma\sigma$. Note that we do not actually need to use market clearing for consumption to pin down the dividend yield; all we need is the excess return, which is pinned down by risk aversion. We already have optimal consumption as a function of returns. Plugging these into the sustainability constraint, we have

$$
\begin{aligned}
&\epsilon\left(\rho + \frac{1}{\epsilon}g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2 - \rho + \frac{\gamma^2\sigma^2}{2\gamma}\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g - \frac{\gamma\left(1+\frac{1}{\epsilon}\right)}{2}\sigma^2 + \frac{\gamma\sigma^2}{2}\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g + \frac{\gamma\sigma^2}{2}\left(1 - 1 - \frac{1}{\epsilon}\right)\right) \geq 0, \\
&\epsilon\left(\frac{1}{\epsilon}g - \frac{\gamma\sigma^2}{2}\left(\frac{1}{\epsilon}\right)\right) \geq 0, \\
&g - \frac{\gamma\sigma^2}{2} \geq 0.
\end{aligned}
$$

B Proof of Proposition 2, Disaster Risk in Section 3.1

From Campbell and Martin (2023), optimal consumption for CRRA utility with disaster risk is

$$\frac{c}{W} = \frac{\rho + (\gamma - 1)(r + \hat{\mu} - \frac{1}{2}\gamma\sigma^2) - \lambda \mathbb{E}\left[X^{1-\gamma} - 1\right]}{\gamma},$$

and Pindyck and Wang (2013) have

$$\frac{c}{W} = \rho + (\gamma - 1)\left(g - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right).$$

The sustainability constraint (SC) with $\alpha = 1$ is

$$SC = r - c/w + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma} - 1\right] \geq 0.$$

Let's break these into pieces. First, we have

$$\begin{aligned} r - c/w = \rho + \gamma g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] \\ - \left(\rho + (\gamma - 1)\left(g - \frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right)\right), \end{aligned}$$

and so

$$\begin{aligned} r - c/w &= g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] - \left((\gamma - 1)\left(-\frac{1}{2}\gamma\sigma^2 - \frac{\lambda}{1-\gamma}\mathbb{E}\left[1 - X^{1-\gamma}\right]\right)\right), \\ &= g - \frac{\gamma(1+\gamma)}{2}\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] + \frac{1}{2}(\gamma - 1)\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right], \\ &= g - \frac{\gamma}{2}\sigma^2(1 + \gamma - (\gamma - 1)) - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right], \\ &= g - \gamma\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right]. \end{aligned}$$

Then we can write $\hat{\mu} - \frac{1}{2}\gamma\sigma^2$ as

$$\begin{aligned} \hat{\mu} - \frac{1}{2}\gamma\sigma^2 &= \gamma\sigma^2 + \lambda\mathbb{E}\left[(1 - X)(X^{-\gamma} - 1)\right] + \lambda\mathbb{E}\left[(1 - X)\right] - \frac{1}{2}\gamma\sigma^2, \\ &= \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[(1 - X)X^{-\gamma}\right]. \end{aligned}$$

Combining these we have

$$
\begin{aligned}
r - c/w + \hat{\mu} - \frac{1}{2}\gamma\sigma^2 &= g - \gamma\sigma^2 - \lambda\mathbb{E}\left[(X^{-\gamma} - 1)\right] + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right] + \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[(1-X)X^{-\gamma}\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma} + (1-X)X^{-\gamma} - (X^{-\gamma} - 1)\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma} + X^{-\gamma} - X^{1-\gamma} - X^{-\gamma} + 1\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + 2\lambda\mathbb{E}\left[1 - X^{1-\gamma}\right].
\end{aligned}
$$

Finally,

$$
\begin{aligned}
SC &= g - \frac{1}{2}\gamma\sigma^2 + 2\lambda\mathbb{E}\left[1 - X^{1-\gamma}\right] + \frac{\lambda}{1-\gamma}\mathbb{E}\left[X^{1-\gamma} - 1\right], \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right]\left(2 - \frac{1}{1-\gamma}\right), \\
&= g - \frac{1}{2}\gamma\sigma^2 + \lambda\mathbb{E}\left[1 - X^{1-\gamma}\right]\left(\frac{1-2\gamma}{1-\gamma}\right).
\end{aligned}
$$

Let X have Power distribution in $(0,1)$ with shape $\beta > 0$. Then $\mathbb{E}[X^n] = \frac{\beta}{\beta+n}$, and so

$$
\mathbb{E}\left[X^{1-\gamma}\right] = \frac{\beta}{\beta + 1 - \gamma},
$$

which means

$$
\mathbb{E}\left[1 - X^{1-\gamma}\right] = \frac{1-\gamma}{\beta + 1 - \gamma},
$$

and therefore we have

$$
\begin{aligned}
SC &= g - \frac{1}{2}\gamma\sigma^2 + \lambda\frac{1-2\gamma}{\beta+1-\gamma}, \\
&= g - \frac{1}{2}\gamma\sigma^2 - \lambda\frac{2\gamma-1}{\beta-(\gamma-1)}.
\end{aligned}
$$

C Results for Long-run Risk in Section 3.2

We first do partial equilibrium with log utility. There are two states, $0, 1$, with $\mu_0 < \mu_1$. The state changes at Poisson rate λ. Hence, state 1 is the good state.

Partial Equilibrium We first solve the standard optimization problem. Let the value function in state i be V_i. Let α_i denote the portfolio and let c_i denote consumption. The investor HJB is

$$\rho V_i = \max_{c,\alpha} \left\{ \log(c) + V_i'(r + \alpha\mu_i - c/W)W + \alpha_i^2 \frac{\sigma^2}{2} W^2 V_i'' + \lambda \left(V_j - V_i\right) \right\}.$$

Proposition 9. *Optimality conditions are $c_i = \rho W$ and $\alpha_i = \frac{\mu_i}{\sigma^2}$. Welfare is given by*

$$V_i = \frac{\log W}{\rho} + \frac{\log(\rho)}{\rho} + \frac{1}{\rho^2}\left(r - \rho + \frac{SR_i^2}{2}\right) - \frac{\lambda}{2\rho^2(\rho + 2\lambda)}\left(SR_i^2 - SR_j^2\right). \tag{32}$$

In particular, the Markov shock does not affect the optimality conditions, but it does affect welfare.

Proof. With log utility, the value function can be written as

$$V_i = \frac{\log W}{\rho} + b_i,$$

so that $V' = \frac{1}{\rho W}$ and $V'' = -\frac{1}{\rho W^2}$. Plugging in, we have

$$\rho V_i = \max_{c,\alpha} \left\{ \log(c) + \frac{1}{\rho}(r + \alpha\mu_i - c/W) - \alpha_i^2 \frac{\sigma^2}{2\rho} + \lambda \left(b_j - b_i\right) \right\},$$

and taking FOCs, we have the standard conditions

$$\frac{1}{c} = \frac{1}{W\rho} \implies c = \rho W, \quad \frac{1}{\rho}\mu_i = \alpha_i \frac{\sigma^2}{\rho} \implies \alpha = \frac{\mu_i}{\sigma^2}. \tag{33}$$

Importantly, the Markov shock does not affect these optimality conditions. The value function therefore satisfies

$$\begin{aligned}
\log W + \rho b_i &= \log(\rho) + \log W + \frac{1}{\rho}(r + SR_i^2 - \rho) - \frac{SR_i^2}{2\rho} + \lambda \left(b_j - b_i\right), \\
\rho b_i &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_i^2}{2\rho} + \lambda \left(b_j - b_i\right), \\
(\rho + \lambda) b_i &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_i^2}{2\rho} + \lambda b_j, \\
(\rho + \lambda) b_j &= \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_j^2}{2\rho} + \lambda b_i.
\end{aligned}$$

Subtracting equations for i, j, we therefore have

$$(\rho+\lambda)(b_i-b_j)=\frac{SR_i^2-SR_j^2}{2\rho}-\lambda(b_i-b_j),$$
$$(\rho+2\lambda)(b_i-b_j)=\frac{SR_i^2-SR_j^2}{2\rho},$$
$$(b_i-b_j)=\frac{SR_i^2-SR_j^2}{2\rho(\rho+2\lambda)}.$$

Hence, we have

$$\rho b_i=\log(\rho)+\frac{1}{\rho}\left(r-\rho+\frac{SR_i^2}{2}\right)-\frac{\lambda}{2\rho(\rho+2\lambda)}\left(SR_i^2-SR_j^2\right).$$

□

General Welfare Function We now characterize welfare for a given θ_i, α_i. Recall that $\nu = r+\alpha\mu-\theta-\alpha^2\frac{\sigma^2}{2}$.

Proposition 10. *For a given θ, α, welfare is given by $V_i = \frac{\log W}{\rho} + b_i$, with*

$$\rho b_i=\log(\theta_i)+\frac{1}{\rho}\nu_i-\frac{\lambda}{\rho+2\lambda}\left(\log(\theta_i)-\log(\theta_j)+\frac{1}{\rho}\left(\nu_i-\nu_j\right)\right). \tag{34}$$

Proof. Let the welfare function be given by

$$V_i = A_i\log W + b_i,$$

so that the first coefficient could change with the state. Then we have

$$\rho A_i\log W+\rho b_i=\log(\theta_i)+\log W+A_i\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda\left(b_j-b_i\right).$$

We immediately have $\rho A_i = 1$, as before. Continuing

$$\rho b_i=\log(\theta_i)+\frac{1}{\rho}\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda\left(b_j-b_i\right),$$
$$(\rho+\lambda)b_i=\log(\theta_i)+\frac{1}{\rho}\left(r+\alpha_i\mu_i-\theta_i-\alpha_i^2\frac{\sigma^2}{2}\right)+\lambda b_j,$$

and we also have

$$(\rho+\lambda)b_j = \log(\theta_j) + \frac{1}{\rho}\left(r + \alpha_j\mu_j - \theta_j - \alpha_j^2\frac{\sigma^2}{2}\right) + \lambda b_i.$$

Plugging in for ν_i, we can write

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \lambda\left(b_i - b_j\right). \tag{35}$$

We can combine to get

$$\begin{aligned}(\rho+\lambda)(b_i-b_j) &= \log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i-\nu_j) - \lambda(b_i-b_j),\\ (\rho+2\lambda)(b_i-b_j) &= \log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i-\nu_j).\end{aligned}$$

Hence,

$$(b_i - b_j) = \frac{1}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i-\nu_j\right)\right). \tag{36}$$

Plugging in, we therefore have

$$\begin{aligned}\rho b_i &= \log(\theta_i) + \frac{1}{\rho}\left(\nu_i\right) - \frac{\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i-\nu_j\right)\right),\\ &= \left(\frac{\rho+\lambda}{\rho+2\lambda}\right)\left(\log(\theta_i) + \frac{1}{\rho}\left(\nu_i\right)\right) + \frac{\lambda}{\rho+2\lambda}\left(\log(\theta_j) + \frac{1}{\rho}\left(\nu_j\right)\right).\end{aligned}$$

At the optimal, we have

$$\nu_i = r - \rho + \frac{1}{2}SR_i^2.$$

□

Adding the Sustainability Constraint We now impose that welfare cannot have a negative drift. The expected change in welfare is the drift in wealth minus the Itô term reflecting risk plus the Poisson term.

$$\begin{aligned}\mathbb{E}[\mathrm{d}V_i] &= \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i-b_j),\\ &= \frac{\nu_i}{\rho} - \lambda(b_i-b_j).\end{aligned} \tag{37}$$

Proof of Proposition 3. Let's first go directly to the optimized conditions, in which case $\theta_i = \theta_j = \rho$

and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. Using equation (20), we can write

$$\mathbb{E}[\mathrm{d}V_i] = r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+2\lambda)}\left(SR_i^2 - SR_j^2\right).$$

□

When the constraint binds, we then have the following condition on θ_i, α_i:

$$r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2 = \rho\lambda(b_i - b_j), \implies \nu_i = \rho\lambda(b_i - b_j). \tag{38}$$

Because $b_i - b_j$ is a function of θ_i, α_i from (20), we cannot solve for θ_i explicitly.

Proposition 11. *When the SC binds, the optimal portfolio is unchanged, $\alpha_i = \frac{\mu_i}{\sigma^2}$, and consumption solves*

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right). \tag{39}$$

This means that $\theta < \rho$ (i.e., we need a lower θ to satisfy the constraint).

Proof. By solving for $b_i - b_j$ from the sustainability constraint, we can plug into equation (35) to write b_i as

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{1}{\rho}(\nu_i) = \log(\theta_i).$$

Maximizing welfare comes down to maximizing θ subject to the sustainability constraint. The difference is that our sustainability constraint is not so simple, as noted earlier. Use $\nu_i = \rho\lambda(b_i - b_j)$ from the sustainability constraint with equation (20):

$$\begin{aligned}
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i - \nu_j)\right), \\
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right) + \frac{\lambda}{\rho+2\lambda}\nu_i, \\
\nu_i\left(1 - \frac{\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i\left(\frac{\rho+2\lambda-\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i\left(\frac{\rho+\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i &= \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right).
\end{aligned}$$

This means we have

$$r + \alpha_i \mu_i - \theta_i - \frac{1}{2}\alpha_i^2 \sigma^2 = \frac{\rho\lambda}{\rho+\lambda} \left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),$$

or

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda} \log(\theta_i) = r + \alpha_i \mu_i - \frac{1}{2}\alpha_i^2 \sigma^2 + \frac{\rho\lambda}{\rho+\lambda} \left(\log(\theta_j) + \nu_j/\rho\right). \tag{40}$$

Recall that we want to maximize $\log\theta$ (i.e., maximize θ) subject to the constraint. The LHS is increasing in θ, which means we want to maximize the RHS. Hence,

$$\alpha_i = \frac{\mu_i}{\sigma^2},$$

the same FOC from earlier. The portfolio decision is not distorted, same as before. Thus, the consumption rate implicitly solves

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda} \log(\theta_i) = r + \frac{1}{2} SR_i^2 + \frac{\rho\lambda}{\rho+\lambda} \left(\log(\rho) + (r - \rho + \frac{1}{2} SR_j^2)/\rho\right), \tag{41}$$

where we have substituted in state j. □

General Equilibrium

Proof of Proposition 4. We need to characterize the wedge $\rho\lambda(b_i - b_j)$ in GE. From equation (20), we have

$$\rho\lambda(b_i - b_j) = \frac{\rho\lambda}{\rho+2\lambda} \left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}\left(\nu_i - \nu_j\right)\right).$$

Thus, in GE and plugging in optimal consumption we have

$$\rho\lambda(b_i - b_j) = \frac{\lambda}{\rho+2\lambda} \left(g_i - g_j\right).$$

Thus, sustainability will bind in 1 if $g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+2\lambda}\left(g_1 - g_0\right) < 0$. □

There are two things to note here. First, if g_0 is sufficiently low, then the constraint will bind in 0, which means that optimal consumption will not be ρ—and this will affect the constraint. Second, it's plausible that λ is not very high; perhaps there is a 2% chance of a disaster in a given year. The growth loss $g_1 - g_0$ has to be substantial for this to matter. With one-sided risk (see the next section) the denominator will be $\rho + \lambda$ instead of $\rho + 2\lambda$, which will increase the cost, making the SC more likely to bind. It takes substantial growth risks to make this bind because λ is not 1.

C.1 Downside Risk: Low Growth Absorbing State

Now suppose that we start in the good state and we switch to the bad state Poisson rate λ. But now we think of the bad state as permanent disaster (e.g., climate risk)—and so once, we move to the bad state we stay there.

To start, let's suppose the SC constraint does NOT bind in the bad state so we can do a standard optimization. (The GE analysis might suggest otherwise.) The previous optimization goes through. We update the value functions as follows. We update the value function as follows:

Partial Equilibrium

Proposition 12. *With downside (absorbing) risk, the state-*0 *welfare intercept is given simply by*

$$\rho b_0 = \log(\rho) + \frac{1}{\rho}(r - \rho) + \frac{SR_0^2}{2\rho},$$

*and the state-*1 *welfare intercept is*

$$\rho b_1 = \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) - \frac{\lambda}{2\rho(\rho + \lambda)}\left(SR_1^2 - SR_0^2\right).$$

Proof. First, the state-0 function follows immediately because there is no state risk. The state-1 function is

$$\begin{aligned}
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) - \lambda\left(b_1 - b_0\right), \\
(\rho + \lambda) b_1 &= \log(\rho) + \frac{1}{\rho}\left(r - \rho + \frac{SR_1^2}{2}\right) + \lambda b_0.
\end{aligned}$$

Plugging in we have

$$\begin{aligned}
(\rho+\lambda)b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{\lambda}{\rho}\left(\log(\rho)+\frac{1}{\rho}(r-\rho)+\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{\lambda}{\rho}\left(\log(\rho)+\frac{1}{\rho}(r-\rho)+\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho)\left(1+\frac{\lambda}{\rho}\right) + \frac{1}{\rho}(r-\rho)\left(1+\frac{\lambda}{\rho}\right) + \frac{SR_1^2}{2\rho} + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\right),\\
(\rho+\lambda)b_1 &= \log(\rho)\left(\frac{\rho+\lambda}{\rho}\right) + \frac{1}{\rho}(r-\rho)\left(\frac{\rho+\lambda}{\rho}\right) + \frac{SR_1^2}{2\rho} + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}(r-\rho) + \frac{SR_1^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}\right) + \frac{\lambda}{\rho}\left(\frac{SR_0^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}\right)\right).
\end{aligned}$$

Let's rearrange this so it looks more like what we had earlier. We have

$$\begin{aligned}
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) + \frac{SR_1^2}{2\rho}\left(\frac{\rho}{\rho+\lambda}-1\right) + \frac{\lambda}{\rho+\lambda}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \frac{SR_1^2}{2\rho}\left(\frac{\lambda}{\rho+\lambda}\right) + \frac{\lambda}{\rho+\lambda}\left(\frac{SR_0^2}{2\rho}\right),\\
\rho b_1 &= \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \frac{\lambda}{2\rho(\rho+\lambda)}\left(SR_1^2 - SR_0^2\right).
\end{aligned}$$

Compared to earlier, the last term has $\rho+\lambda$ in the denominator, instead of $\rho+2\lambda$. □

For later, it is convenient to write

$$\lambda\left(b_1-b_0\right) = \log(\rho) + \frac{1}{\rho}\left(r-\rho+\frac{SR_1^2}{2}\right) - \rho b_1, \tag{42}$$

$$= \frac{\lambda}{2\rho(\rho+\lambda)}\left(SR_1^2 - SR_0^2\right). \tag{43}$$

General Welfare Function We now characterize welfare for given θ_i, α_i. Recall that $\nu = r + \alpha\mu - \theta - \alpha^2\frac{\sigma^2}{2}$.

Proposition 13. *With downside (absorbing) risk, for a given θ, α, welfare is given by $V_i = \frac{\log W}{\rho} + b_i$, with*

$$\rho b_0 = \log(\theta_0) + \frac{1}{\rho}\nu_0,$$

(no long-run risk) and

$$\rho b_1 = \log(\theta_1) + \frac{1}{\rho}\nu_1 - \frac{\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1 - \nu_0\right)\right). \tag{44}$$

Proof. The welfare function in state 0 follows immediately from previous analysis. It remains to characterize the additional term in state 1. As we did earlier, we can write

$$\begin{aligned}
\rho b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \lambda\left(b_0 - b_1\right), \\
(\rho+\lambda) b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \lambda b_0.
\end{aligned}$$

We plug in $\rho b_0 = \log(\theta_0) + \frac{1}{\rho}\nu_0$ to get

$$\begin{aligned}
(\rho+\lambda) b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 + \frac{\lambda}{\rho}\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= \rho\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= \rho\log(\theta_1) + \lambda\log(\theta_1) - \lambda\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \nu_1 + \frac{\lambda}{\rho}\nu_1 - \frac{\lambda}{\rho}\nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}\nu_0\right), \\
(\rho+\lambda)\rho b_1 &= (\rho+\lambda)\log(\theta_1) + \frac{\rho+\lambda}{\rho}\nu_1 + \lambda\left(\log(\theta_0) - \log(\theta_1) + \frac{1}{\rho}(\nu_0 - \nu_1)\right), \\
\rho b_1 &= \log(\theta_1) + \frac{1}{\rho}\nu_1 - \frac{\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}(\nu_1 - \nu_0)\right).
\end{aligned}$$

□

Using $\lambda(b_1 - b_0) = \log(\theta_1) + \frac{1}{\rho}\nu_1 - \rho b_1$, it is useful to write

$$(b_1 - b_0) = \frac{1}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1 - \nu_0\right)\right). \tag{45}$$

Adding the Sustainability Constraint The analysis with the sustainability constraint in partial equilibrium is no different from in the previous section. What changes is just the welfare weighting of the Markov risk, as already noted. The expected change in welfare is the drift in wealth minus the Itô term reflecting risk plus the Poisson term.

$$\mathbb{E}[\mathrm{d}V_i] = \frac{1}{\rho}\left(r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2\right) - \lambda(b_i - b_j) = \frac{\nu_i}{\rho} - \lambda(b_i - b_j).$$

Proposition 14. *The sustainability constraint will bind whenever*

$$r + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+\lambda)}\left(SR_i^2 - SR_j^2\right) < \rho. \tag{46}$$

Proof. Let's first go directly to the optimized conditions, in which case $\theta_i = \theta_j = \rho$ and $\nu_i - \nu_j = \frac{1}{2}(SR_i^2 - SR_j^2)$. From equation (43)

$$\mathbb{E}[\mathrm{d}V_i] = r - \rho + \frac{1}{2}SR_i^2 - \frac{\lambda}{2(\rho+\lambda)}\left(SR_i^2 - SR_j^2\right).$$

□

When the constraint binds, we then have the following condition on θ_i, α_i:

$$r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2 = \rho\lambda(b_i - b_j), \implies \nu_i = \rho\lambda(b_i - b_j). \tag{47}$$

Proposition 15. *When the SC binds, the optimal portfolio is unchanged, $\alpha_i = \frac{\mu_i}{\sigma^2}$, and consumption solves*

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right). \tag{48}$$

This means that $\theta < \rho$ (i.e., we need a lower θ to satisfy the constraint).

Proof. By solving for $b_i - b_j$ from the sustainability constraint, we can plug into equation (35) to write b_i as

$$\rho b_i = \log(\theta_i) + \frac{1}{\rho}\nu_i - \frac{1}{\rho}(\nu_i) = \log(\theta_i).$$

Maximizing welfare comes down to maximizing θ, subject to the sustainability constraint. The difference is that our sustainability constraint is not so simple, as noted above. Use $\nu_i = \rho\lambda(b_i - b_j)$

from the sustainability constraint with equation (45):

$$\begin{aligned}
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) + \frac{1}{\rho}(\nu_i - \nu_j)\right), \\
\nu_i &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right) + \frac{\lambda}{\rho+2\lambda}\nu_i, \\
\nu_i\left(1 - \frac{\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i\left(\frac{\rho+2\lambda-\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i\left(\frac{\rho+\lambda}{\rho+2\lambda}\right) &= \frac{\rho\lambda}{\rho+2\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right), \\
\nu_i &= \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right).
\end{aligned}$$

This means we have

$$r + \alpha_i\mu_i - \theta_i - \frac{1}{2}\alpha_i^2\sigma^2 = \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_i) - \log(\theta_j) - \nu_j/\rho\right),$$

or

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \alpha_i\mu_i - \frac{1}{2}\alpha_i^2\sigma^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_j) + \nu_j/\rho\right). \tag{49}$$

Recall that we want to maximize $\log\theta$ (i.e., maximize θ), subject to the constraint. The LHS is increasing in θ, which means we want to maximize the RHS. Hence,

$$\alpha_i = \frac{\mu_i}{\sigma^2},$$

the same FOC from earlier. The portfolio decision is not distorted, same as before. Thus, the consumption rate implicitly solves

$$\theta_i + \frac{\rho\lambda}{\rho+\lambda}\log(\theta_i) = r + \frac{1}{2}SR_i^2 + \frac{\rho\lambda}{\rho+\lambda}\left(\log(\rho) + (r - \rho + \frac{1}{2}SR_j^2)/\rho\right), \tag{50}$$

where we have substituted in state j. □

General Equilibrium

Proof of Proposition 5. We need to characterize the wedge $\rho\lambda(b_i - b_j)$ in GE. From equation (44), we have

$$\rho\lambda(b_1 - b_0) = \frac{\rho\lambda}{\rho+\lambda}\left(\log(\theta_1) - \log(\theta_0) + \frac{1}{\rho}\left(\nu_1 - \nu_0\right)\right).$$

Thus, in GE and plugging in optimal consumption, we have

$$\rho\lambda(b_1 - b_0) = \frac{\lambda}{\rho+\lambda}\left(g_1 - g_0\right).$$

Thus, sustainability will bind in 1 if $g_1 - \frac{1}{2}\sigma^2 - \frac{\lambda}{\rho+\lambda}\left(g_1 - g_0\right) < 0$. □