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On the Optimal Wealth Process in a Log-Normal Market:

Applications to Risk Management

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On the optimal wealth process in a log-normal market: Applications to risk management*

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Abstract

The theory of portfolio choice holds that investors balance risk and reward in their investment decisions. We explore the relationship between investors' attitudes towards taking risk and their objectives for managing the risk they take on. Working in a classical theoretical model, we calculate the distribution and density functions of an investor's optimal wealth process and prove new mathematical results for these functions under general risk preferences. By applying our results to a constant relative risk aversion investor who has a targeted value at risk or expected shortfall at a given future time, we are able to infer the investor's risk preferences and prescribe how to invest to achieve the desired goal. Then, drawing analogies to the option greeks, we define and derive closed-form expressions for "portfolio greeks," which measure the sensitivities of an investor's optimal wealth to changes in the cumulative excess stock return, time, and market parameters. Like option greeks, portfolio greeks can be used in the risk management of investors' portfolios.

Keywords: expected utility; Merton problem; value at risk (VaR); expected shortfall; portfolio greeks

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1 Introduction

This paper contributes to investment management in a log-normal market by providing a study of the optimal wealth process in the classical Merton problem on a finite trading horizon. The analysis is based on a stochastic representation (22) of the optimal wealth process in terms of a space-time harmonic function of the underlying Brownian motion, market price of risk, and investor's initial wealth. A similar stochastic representation exists for the optimal portfolio process (see (23)). These representations were first derived in Musiela and Zariphopoulou (2010) for an alternative class of risk preferences (the so-called forward investment performance processes) and were recently used in Källblad and Zariphopoulou (2014) to study qualitative properties of the optimal portfolio process in a multi-asset log-normal model.

We use the stochastic representations to study the optimal wealth and portfolio processes in more detail. First, we derive novel and explicit representation formulae for the optimal wealth and portfolio processes across different utilities. We show that, for two arbitrary utilities and with modified initial conditions, the associated optimal wealth processes can be written in terms of one another using a deterministic function that solves a linear parabolic problem (see (30)–(32)). The analogous transformation for the respective optimal portfolio processes is similar (see (33)).

Next, we use the stochastic representations to compute the cumulative distribution and density functions of the optimal wealth process at a fixed time. We show that these functions can be expressed in terms of the space-aggregate local absolute risk aversion and the time-aggregate marginal local absolute risk tolerance of the investor (see (38) and (39)). For general utilities, we examine how properties of the absolute risk tolerance coefficient affect, for all intermediate times, the behavior of the cumulative distribution function and, in particular, the probability of the optimal wealth falling below the initial wealth. For the former, we derive universal bounds that depend only on bounds of the slope of the absolute risk tolerance coefficient (see (43) and (44)). For the latter, we show that it is exclusively the concavity or convexity of risk tolerance that determines the monotonicity of the probability of falling below initial wealth (see (47)). Using the expressions for the cumulative distribution and density functions, we represent the expectation of functionals of the optimal wealth process at a fixed time in terms of a convolution evaluated at a specific point (see (51)). As an application, we compute the mean and variance of the optimal wealth process at a fixed time.

We continue our study of the optimal wealth process by considering applications to risk management. We use the stochastic representation of the optimal wealth to express its quantile function at a fixed time in terms of the associated harmonic function (see (57)), which, in turn, is used to produce explicit representation formulae for the investor's value at risk (VaR) and expected shortfall (ES) of his optimal wealth (see (59) and (64)). We then look at the interplay between the investor's risk preferences and the investment targets he sets, building on work initiated in Musiela and Zariphopoulou (2010) and Monin (2014). We study cases in which the investor sets a target for the VaR or the ES of his optimal wealth. Specifically, we examine how to infer from such targets the investor's relative risk aversion parameter for CRRA utilities. We show that, for a CRRA investor, a single VaR target at any intermediate trading time uniquely determines the investor's implied relative risk aversion coefficient, for which we produce an explicit formula (see (61)). We also discuss inferring the relative risk aversion for a CRRA investor who targets the ES of his wealth, showing that, under a mild additional assumption, the investor's relative risk aversion coefficient can be found by numerical inversion.

Finally, we examine sensitivities of the optimal wealth process, drawing analogies between them and similar quantities in derivatives. For this, we first take the stock's cumulative excess return, rather than its level, as the natural "underlying" and express the optimal wealth process as a deterministic function of this underlying (see (69)). In analogy to option greeks, we then define "portfolio greeks" and provide explicit representation formulae in terms of the investor's marginal local absolute risk tolerance, his optimal wealth and portfolio processes, and the stock's cumulative excess return (see (80), (81), (87)–(89)). We conclude by deriving sensitivities for the cumulative excess return on the optimal wealth, rather than its level, and show how these sensitivities relate to the beta of the investor's portfolio.

The paper is organized as follows. We introduce the model in section 2. In section 3, we recall the stochastic representations for the optimal wealth and portfolio processes and consider these processes across different utilities. In section 4, we provide results on the cumulative distribution and density functions, study the probability of falling below the investor's initial wealth, and provide a representation result for the expectation of a functional of the optimal wealth at a fixed time. In section 5, we study the quantile function of the optimal wealth at a fixed time and, in turn, the VaR and expected shortfall of the optimal wealth. Finally, in section 6 we analyze the sensitivities of the optimal wealth process.

2 The model and its optimal wealth and portfolio processes

We briefly recall the classical Merton problem (Merton (1969)), its value function and its solution. Trading takes place in $[0,T]$, with the horizon T being arbitrary but fixed. The market environment consists of one riskless asset and one risky stock, whose price, S_t, $t \geq 0$, is modeled as a log-normal process,

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right), \tag{1}$$

with $S_0 > 0$. The process W_t, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The underlying filtration is taken to be $\mathcal{F}_t = \sigma\left(W_s : 0 \leq s \leq t\right)$. The coefficients μ and σ are positive constants. The riskless asset, the savings account, offers constant interest rate $r > 0$. We denote

$$\lambda = \frac{\mu - r}{\sigma}, \tag{2}$$

and we assume, without loss of generality, that $\lambda > 0$.

Starting at $t \in [0,T)$ with initial wealth $x>0$, the investor invests at any time $s \in (t,T]$ in the riskless and risky assets. The present value of the amounts invested are denoted, respectively, by π_s^0 and π_s. The present value of his investment is then given by $X_s^\pi = \pi_s^0 + \pi_s$, $s \in (t,T]$. We will refer to X_s^π as the discounted wealth generated by the strategy $\left(\pi_s^0, \pi_s\right)$. The investment strategies play the role of control processes and are taken to be self-financing. We easily deduce that the discounted wealth satisfies

$$dX_s^\pi = \sigma \pi_s \left(\lambda ds + dW_s\right), \tag{3}$$

for $s \in (t,T]$, with initial wealth $X_t = x$. An investment process π_s, $s \in [t,T]$, is admissible if $\pi_s \in \mathcal{F}_s$, $E\left(\int_t^T \pi_s^2 ds\right) < \infty$ and the associated wealth remains non-negative, $X_s^\pi \geq 0$, $s \in [t,T]$. We denote the set of admissible strategies by $\mathcal{A}$.

The investor's utility function at T is given by $U\colon \mathbb{R}_+ \to \mathbb{R}_+$, and is assumed to be a strictly concave, strictly increasing and $C^4(0,\infty)$ function, satisfying the standard Inada conditions

$$\lim_{x\to 0} U'(x) = \infty \quad \text{and} \quad \lim_{x\to\infty} U'(x) = 0. \tag{4}$$

We recall the inverse, $I\colon \mathbb{R}_+ \to \mathbb{R}_+$, of the marginal utility U',

$$I(x) = \left(U'\right)^{(-1)}(x), \tag{5}$$

and assume that, for some $\varepsilon > 0$, it satisfies the polynomial growth condition,

$$I(x) \leq \varepsilon + x^{-\varepsilon}. \tag{6}$$

The value function is then defined as the maximal expected utility,

$$u(x,t) = \sup_{\pi \in \mathcal{A}} E\left(U\left(X_T^\pi\right) \middle| X_t^\pi = x\right), \tag{7}$$

where X_s^π, $s \in [t,T]$, solves (3).

The above stochastic optimization problem has been widely studied and completely solved. We provide the main results below without proof (see, for example, Karatzas et al. (1987) and Björk (2009)).

Proposition 2.1. *i) The value function $u \in C^{4,1}\left(\mathbb{R}_+ \times [0,T]\right)$ is strictly increasing and strictly concave in the spatial variable, and solves the Hamilton-Jacobi-Bellman (HJB) equation,*

$$u_t - \frac{1}{2}\lambda^2 \frac{u_x^2}{u_{xx}} = 0, \tag{8}$$

with $u(x,T) = U(x)$ and λ as in (2).

ii) The optimal portfolio process is given, for $s \in [t,T]$, by

$$\pi_s^{*,x} = \pi^*\left(X_s^{*,x}, s\right), \tag{9}$$

where the optimal feedback portfolio function $\pi^* : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ is given by

$$\pi^*(x,t) = -\frac{\lambda}{\sigma} \frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{10}$$

with $X_s^{*,x}$, $s \in [t,T]$, *being the optimal wealth process solving* (3) *with* $\pi_s^{*,x}$ *as in (9).*

Associated with any utility function are the *(absolute) risk tolerance coefficient* and the *(absolute) risk aversion coefficient*, denoted respectively by the functions $RT(x)$ and $RA(x)$, and given, for $x > 0$, by

$$RT(x) = -\frac{U'(x)}{U''(x)} \qquad \text{and} \qquad RA(x) = -\frac{U''(x)}{U'(x)}. \tag{11}$$

We assume that the risk tolerance coefficient $RT(x)$ is strictly increasing for $x > 0$ and satisfies $R(0) := \lim_{x \downarrow 0} RT(x) = 0$ (see, among others, Xia (2011) and Källblad and Zariphopoulou (2014)).

For intermediate trading times $t \in [0,T)$, one then defines the associated *local,* or *indirect,* absolute coefficients. The *local (absolute) risk tolerance,* $r(x,t)$, and the *local (absolute) risk aversion,* $\gamma(x,t)$, are given, respectively, by

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)} \quad \text{and} \quad \gamma(x,t) = -\frac{u_{xx}(x,t)}{u_x(x,t)}, \tag{12}$$

with u being the value function (7). Therefore (cf. (10) and (12)), the optimal portfolio process, π_s^*, is given, for $s \in [t,T]$, by

$$\pi_s^{*,x} = \frac{\lambda}{\sigma} r\left(X_s^{*,x}, s\right). \tag{13}$$

3 The optimal wealth process

We review the representation results for the optimal wealth and portfolio processes used in Källblad and Zariphopoulou (2014). As (22) and (24) show, these processes are represented as harmonic functionals of the current value of the Brownian motion that drives the stock price process.

Such representations were first obtained by Musiela and Zariphopoulou (2010) under forward investment performance criteria and general Itô price processes. We remark that in Musiela and Zariphopoulou (2010), the transformation analogous to (14) does not involve the terminal horizon, since forward criteria are defined for all trading times. Therein, however, time is rescaled while herein it is not, as rescaling time would have resulted in an artificially altered terminal investment horizon.

We start with some preliminary results (see Källblad and Zariphopoulou (2014, Proposition 4)).

Proposition 3.1. *Let* $I : \mathbb{R}_+ \to \mathbb{R}_+$ *be given by* (5) *and assume that it satisfies the growth condition* (6). *Let* $H : \mathbb{R} \times [0,T] \to \mathbb{R}_+$ *be defined by*

$$u_x\left(H(x,t),t\right) = \exp\left(-x - \frac{1}{2}\lambda^2 (T-t)\right), \tag{14}$$

where $u(x,t)$ *is the value function (cf.* (7)*) and* λ *is as in* (2). *Then, the following assertions hold.*

i) The function $H(x,t)$ solves the heat equation

$$H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \tag{15}$$

with terminal condition

$$H(x,T) = I\left(e^{-x}\right). \tag{16}$$

ii) For each $t \in [0,T]$, the function $H(x,t)$ is strictly increasing in x and is of full range, that is, $\lim_{x\to-\infty} H(x,t) = 0$ and $\lim_{x\to\infty} H(x,t) = \infty$.

iii) The local absolute risk tolerance function $r \in C^{2,1}(\mathbb{R}_+ \times (0,T])$ satisfies

$$r(x,t) = H_x\left(H^{(-1)}(x,t), t\right), \tag{17}$$

where $H(x,t)$ solves (15) and (16).

The following proposition provides results on equations that the spatial inverse $H^{(-1)}$ satisfies as well as on the representation of its temporal and spatial increments.

Proposition 3.2. *The spatial inverse $H^{(-1)} \colon \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ satisfies*

$$H_t^{(-1)}(x,t) = \frac{1}{2}\lambda^2 r_x(x,t) \tag{18}$$

and

$$H_x^{(-1)}(x,t) = \gamma(x,t), \tag{19}$$

where $r(x,t)$ and $\gamma(x,t)$ are as in (12). Therefore, the temporal and spatial increments of $H^{(-1)}$ can be written as

$$H^{(-1)}(x,t) - H^{(-1)}(x,0) = \frac{1}{2}\lambda^2 \int_0^t r_x(x,s)\, ds \tag{20}$$

and

$$H^{(-1)}(y,t) - H^{(-1)}(x,t) = \int_x^y \gamma(z,t)\, dz. \tag{21}$$

Proof. Assertion (18) follows from

$$H_t^{(-1)}(x,t) = -\frac{H_t\left(H^{(-1)}(x,t),t\right)}{H_x\left(H^{(-1)}(x,t),t\right)}$$

$$= \frac{1}{2}\lambda^2 \frac{H_{xx}\left(H^{(-1)}(x,t),t\right)}{H_x\left(H^{(-1)}(x,t),t\right)} = \frac{1}{2}\lambda^2 r_x(x,t),$$

where we used (15) and (17). The rest of the proof follows easily. □

Next, we provide the stochastic representations for the optimal wealth and portfolio processes. For its proof see Musiela and Zariphopoulou (2010) and Källblad and Zariphopoulou (2014).[1] For convenience, we state all results throughout assuming that the initial time $t = 0$.

[1] As is mentioned in Källblad and Zariphopoulou (2014), one could use standard duality results to derive (22) and (24). The construction is in reverse order, in that the wealth representation (22) is established first and (23) then follows from a direct application of Itô's formula and (3).

Proposition 3.3. *The optimal wealth* $X_t^{*,x}$, $t \in [0,T]$, *starting at* x *at time* 0, *and the associated optimal portfolio* $\pi_t^{*,x}$ *are given, respectively, by the processes*

$$X_t^{*,x} = H\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right) \tag{22}$$

and

$$\pi_t^{*,x} = \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(X_t^{*,x}, t), t\right) \tag{23}$$

$$= \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right), \tag{24}$$

where the function H *satisfies* (15) *and* (16).

Example 1 (CRRA utility)**.** Let $U(x)$ be given by

$$U(x) = \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma}, & \gamma > 0,\ \gamma \neq 1, \\ \log x, & \gamma = 1. \end{cases} \tag{25}$$

Then, the function $I(x)$ (cf. (5)) is given by $I(x) = x^{-1/\gamma}$ and, in turn,

$$H(x,t) = \exp\left(\frac{x}{\gamma} + \frac{1}{2}\frac{\lambda^2}{\gamma^2}(T-t)\right). \tag{26}$$

Using that $H^{(-1)}(x,0) = \gamma \log y - \frac{1}{2}\frac{\lambda^2}{\gamma}T$ and $H_x(x,t) = \frac{1}{\gamma}H(x,t)$, (22) and (23) give the familiar formulae

$$X_t^{*,x} = x \exp\left(\frac{\lambda^2}{\gamma}\left(1 - \frac{1}{2\gamma}\right)t + \frac{\lambda}{\gamma}W_t\right) \tag{27}$$

and

$$\pi_s^{*,x} = \frac{\lambda}{\sigma\gamma} X_t^{*,x}. \tag{28}$$

Example 2. The results in Example 1 can be easily generalized when the inverse marginal function $I(x)$ (cf. (5)) is given by $I(x) = \sum_{i=1}^N x^{-1/\gamma_i}$, $\gamma_i > 0, i = 1, \ldots, N$. Then, we find that

$$H(x,t) = \sum_{i=1}^N \exp\left(\frac{x}{\gamma_i} + \frac{1}{2}\frac{\lambda^2}{\gamma_i^2}(T-t)\right),$$

$X_t^{*,x} = \sum_{i=1}^N e^{N(x,t;T)}$ and $\pi_s^{*,x} = \frac{\lambda}{\sigma}\sum_{i=1}^N \frac{1}{\gamma_i} e^{N(x,t;T)}$, where $N(x,t;T) = \frac{H^{(-1)}(x,0)}{\gamma_i} + \frac{\lambda^2}{\gamma_i}(1 - \frac{1}{2\gamma_i})t + \lambda W_t + \frac{1}{2}\frac{\lambda^2}{\gamma_i^2}T$. Note though that analogous results do not hold when $U'(x) = \sum_{i=1}^N x^{-\gamma_i}$, $\gamma_i > 0, i = 1, \ldots, N$.

3.1 Optimal wealth processes across different utilities

Using the stochastic formulae (22) and (23) we are able to associate the optimal wealth and portfolio processes corresponding to different utility functions. Specifically, the optimal processes for two arbitrary utility functions can be expressed as a deterministic transformation of each other with appropriately modified initial wealths.

Proposition 3.4. *Let U and $\widehat{U}$ be two utility functions and $g:\mathbb{R}_+\to\mathbb{R}_+$ be defined by*

$$U'(x)=\widehat{U}'(g(x)). \tag{29}$$

i) Let $X_t^{*,x}$ and $\widehat{X}_t^{*,x}$ be the associated wealth processes, starting at wealth x at time $t=0$. Let $r(x,t)$ be the local absolute risk tolerance function associated with utility U (cf. (12)). Then,

$$\widehat{X}_t^{*,x}=G\left(X_t^{*,G^{(-1)}(x,0)},t\right), \tag{30}$$

where the function $G\colon \mathbb{R}_+\times[0,T]\to\mathbb{R}_+$ satisfies

$$G_t(x,t)+\frac{1}{2}\lambda^2r^2(x,t)\,G_{xx}(x,t)=0 \tag{31}$$

with terminal condition

$$G(x,T)=g(x). \tag{32}$$

ii) Let $\pi_t^{*,x}$ and $\widehat{\pi}_t^{*,x}$ be the associated optimal portfolio processes and let $\pi^*(x,t)$ and $\widehat{\pi}^*(x,t)$ be the corresponding optimal feedback portfolio functions. Then,

$$\widehat{\pi}_t^{*,x}=G_x\left(X_t^{*,G^{(-1)}(x,0)},t\right)\pi_t^{*,G^{(-1)}(x,0)} \tag{33}$$

and

$$\widehat{\pi}^*(x,t)=G_x(G^{(-1)}(x,t),t)\pi^*\left(G^{(-1)}(x,t),t\right). \tag{34}$$

Proof. i) First note that the function g is well defined, since $g(x)=\widehat{I}(U'(x))$, with $\widehat{I}=\left(\widehat{U}'\right)^{(-1)}$. Next, let $u(x,t)$ and $\widehat{u}(x,t)$ be the value functions (cf. (7)) corresponding to utilities $U(x)$ and $\widehat{U}(x)$, and $H(x,t)$ and $\widehat{H}(x,t)$ be the associated harmonic functions, defined in (14). Then, for $h=H^{(-1)},\widehat{H}^{(-1)}$ and $v_x=u_x,\widehat{u}_x$, we have

$$h(x,t)=-\log v_x(x,t)-\frac{1}{2}\lambda^2(T-t). \tag{35}$$

Define $G\colon\mathbb{R}_+\times[0,T]\to\mathbb{R}_+$ such that, for $t\in[0,T]$,

$$u_x(x,t)=\widehat{u}_x(G(x,t),t). \tag{36}$$

We have that $G(x,T)=g(x)$ and, for $t\in[0,T)$, $G(x,t)$ is well defined due to the invertibility of $\widehat{u}(x,t)$ in the spatial variable. From (35) we deduce that

$$\widehat{H}(x,t)=G(H(x,t),t). \tag{37}$$

In turn,

$$\widehat{H}_t(x,t) + \frac{1}{2}\lambda^2 \widehat{H}_{xx}(x,t) = G_x(H,t)\left(H_t(x,t) + \frac{1}{2}\lambda^2 H_{xx}(x,t)\right)$$

$$+G_t(H,t) + \frac{1}{2}\lambda^2 H_x^2(x,t)\, G_{xx}(H,t)\,.$$

Because both $H(x,t)$ and $\widehat{H}(x,t)$ solve (15), we obtain

$$G_t(x,t) + \frac{1}{2}\lambda^2 H_x^2\left(H^{(-1)}(x,t),t\right) G_{xx}(x,t) = 0,$$

and using (17) we deduce (31). From the stochastic representation (22) and (37), we then have

$$\begin{aligned}\widehat{X}_t^{*,x} &= \widehat{H}\left(\widehat{H}^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right)\\ &= G\left(H\left(\widehat{H}^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right), t\right)\\ &= G\left(H\left(H^{(-1)}\left(G^{(-1)}(x,0),0\right) + \lambda^2 t + \lambda W_t, t\right), t\right)\\ &= G\left(X_t^{*,G^{(-1)}(x,0)}, t\right).\end{aligned}$$

ii) For the corresponding risk tolerance functions, $r(x,t)$ and $\widehat{r}(x,t)$, we obtain

$$\begin{aligned}\widehat{r}(x,t) &= \widehat{H}_x(\widehat{H}^{(-1)}(x,t),t)\\ &= G_x(H(\widehat{H}^{(-1)}(x,t),t),t)H_x(\widehat{H}^{(-1)}(x,t),t)\\ &= G_x(G^{(-1)}(x,t),t)H_x(H^{(-1)}(G^{(-1)}(x,t),t),t)\\ &= G_x(G^{(-1)}(x,t),t)\cdot r(G^{(-1)}(x,t),t),\end{aligned}$$

where we used that $\widehat{H}_x(x,t) = G_x(H(x,t),t)H_x(x,t)$. We then have, recalling that $\widehat{X}_t^{*,x} = G\left(X_t^{*,G^{(-1)}(x,0)}, t\right)$,

$$\begin{aligned}\widehat{\pi}_t^{*,x} &= \frac{\lambda}{\sigma}\widehat{r}(\widehat{X}_t^{*,x},t)\\ &= \frac{\lambda}{\sigma}G_x\left(G^{(-1)}\left(\widehat{X}_t^{*,x},t\right),t\right) r\left(G^{(-1)}(\widehat{X}_t^{*,x},t),t\right)\\ &= \frac{\lambda}{\sigma}G_x\left(G^{(-1)}\left(G\left(X_t^{*,G^{(-1)}(x,0)},t\right),t\right),t\right) r\left(G^{(-1)}\left(G\left(X_t^{*,G^{(-1)}(x,0)},t\right),t\right),t\right)\\ &= \frac{\lambda}{\sigma}G_x\left(X_t^{*,G^{(-1)}(x,0)},t\right) r\left(X_t^{*,G^{(-1)}(x,0)},t\right)\\ &= G_x\left(X_t^{*,G^{(-1)}(x,0)},t\right)\pi_t^{*,G^{(-1)}(x,0)}.\end{aligned}$$

□

Example 3. Let U be given by the CRRA utility (25). Then, the optimal wealth process corresponding to arbitrary utility $\widehat{U}$ is given by

$$\widehat{X}_t^{*,x} = G\left(G^{(-1)}(x,0)e^{\frac{\lambda^2}{\gamma}\left(1-\frac{1}{2\gamma}\right)t+\frac{\lambda}{\gamma}W_t}, t\right),$$

with $G : \mathbb{R}_+ \times [0,T) \to \mathbb{R}_+$ solving $G_t(x,t) + \frac{1}{2}\left(\frac{\lambda}{\gamma}\right)^2 x^2 G_{xx}(x,t) = 0$ and $G(x,T) = \widehat{I}(x^{-\gamma})$. Moreover, the associated optimal portfolio process is given by

$$\widehat{\pi}_t^{*,x} = G_x\left(G^{(-1)}(x,0)e^{\frac{\lambda^2}{\gamma}\left(1-\frac{1}{2\gamma}\right)t+\frac{\lambda}{\gamma}W_t}, t\right)\pi^{*,G^{(-1)}(x,0)},$$

with corresponding optimal feedback portfolio function

$$\widehat{\pi}^*(x,t) = \frac{\lambda}{\sigma\gamma}G_x\left(G^{(-1)}(x,t), t\right)G^{(-1)}(x,t).$$

4 Probabilistic properties

We examine various probabilistic properties of the optimal wealth process. We provide novel decompositions of the cumulative distribution and density functions of the optimal wealth at a fixed time, derive universal upper and lower bounds for them, and study in detail the probability of falling below the investor's initial wealth. Finally, we use these representations to express the expectation of a functional of the optimal wealth at a fixed time in terms of a convolution.

4.1 The cumulative distribution and density functions

We provide the cumulative distribution and density functions of the optimal wealth process. These functions are represented in terms of two integrals, one temporal and one spatial, of functionals related to the investor's local risk aversion and the marginal local absolute risk tolerance. These representations are particularly useful because they enable us to construct explicit universal bounds (see Corollary 4.4).

Proposition 4.1. *Let λ be as in* (2) *and let $r(x,t)$ and $\gamma(x,t)$ be, respectively, the local absolute risk tolerance and risk aversion functions (cf.* (12)*). The following assertions hold.*

i) Let Φ be the cumulative distribution function of the standard normal distribution. Then, for $t \in (0,T]$, $x, y > 0$ and $X_0^{*,x} = x$, the cumulative distribution function of the optimal wealth at time t is given by

$$\mathbb{P}\left(X_t^{*,x} \leq y\right) = \Phi\left(\frac{1}{\lambda\sqrt{t}}\int_x^y \gamma(z,t)\,dz + \frac{\lambda}{2\sqrt{t}}\int_0^t r_x(x,s)\,ds - \lambda\sqrt{t}\right). \tag{38}$$

ii) Let ϕ be the density function of the standard normal distribution. Then, for $t \in (0,T]$, $x, y > 0$ and $X_0^{*,x} = x$, the density function of the optimal wealth at time

t is given by

$$f(y,t;x,0)=\frac{1}{\lambda\sqrt{t}}\gamma\left(y,t\right)\phi\left(\frac{1}{\lambda\sqrt{t}}\int_x^y\gamma\left(z,t\right)dz+\frac{\lambda}{2\sqrt{t}}\int_0^t r_x(x,s)ds-\lambda\sqrt{t}\right). \tag{39}$$

Proof. From (22) we have,

$$\begin{aligned}\mathbb{P}\left(X_t^{*,x}\leq y\right)&=\mathbb{P}\left(W_t\leq\frac{H^{(-1)}\left(y,t\right)-H^{(-1)}\left(x,0\right)-\lambda^2t}{\lambda}\right)\\&=\Phi\left(\frac{1}{\lambda\sqrt{t}}\left(H^{(-1)}\left(y,t\right)-H^{(-1)}\left(x,0\right)\right)-\lambda\sqrt{t}\right),\end{aligned} \tag{40}$$

and the result follows from (20) and (21). Assertion (39) follows easily from (38). □

Corollary 4.2. *The optimal terminal wealth $X_T^{*,x}$ satisfies*

$$\mathbb{P}\left(X_T^{*,x}\leq y\right)=\Phi\left(\frac{1}{\lambda\sqrt{T}}\log\frac{U'\left(x\right)}{U'\left(y\right)}+\frac{\lambda}{2\sqrt{T}}\int_0^T r_x\left(x,s\right)ds-\lambda\sqrt{T}\right).$$

Example 4. Let U be given by the CRRA utility (25). Using (26) and (17), we easily deduce that $r(x,t)=\frac{1}{\gamma}x$ and $\gamma(x,t)=\frac{\gamma}{x}$. Then, (38) yields

$$\mathbb{P}(X_t^{*,x}\leq y)=\Phi\left(\frac{\gamma}{\lambda\sqrt{t}}\log\left(\frac{y}{x}\right)+\lambda\sqrt{t}\left(\frac{1}{2\gamma}-1\right)\right).$$

The next result relates the sensitivities of the cumulative distribution function with respect to the spatial variables x and y.

Proposition 4.3. *For fixed (y,t), the cumulative distribution function $\mathbb{P}\left(X_t^{*,x}\leq y\right)$ is decreasing with respect to the initial wealth x while, for fixed (x,t), $\mathbb{P}\left(X_t^{*,x}\leq y\right)$ is increasing with respect to the target level y. In particular,*

$$r\left(x,0\right)\frac{\partial\mathbb{P}\left(X_t^{*,x}\leq y\right)}{\partial x}=-r\left(y,t\right)\frac{\partial\mathbb{P}\left(X_t^{*,x}\leq y\right)}{\partial y}, \tag{41}$$

where $r\left(x,t\right)$ is the local absolute risk tolerance function (cf. (12)).

Proof. The monotonicity assertions follow trivially. Next, let

$$A\left(y,t,x,0\right)=\frac{1}{\lambda\sqrt{t}}\left(H^{(-1)}\left(y,t\right)-H^{(-1)}\left(x,0\right)\right)-\lambda\sqrt{t}.$$

Then, from (40), we deduce that

$$\frac{\partial\mathbb{P}\left(X_t^{*,x}\leq y\right)}{\partial y}=\frac{\frac{\partial}{\partial y}A\left(y,t,x,0\right)}{\frac{\partial}{\partial x}A\left(y,t,x,0\right)}\frac{\partial\mathbb{P}\left(X_t^{*,x}\leq y\right)}{\partial x}.$$

In turn, using (19) and (12) yields

$$\frac{\partial}{\partial y} A\left(y,t,x,0\right) = \frac{1}{\lambda\sqrt{t}} H_x^{(-1)}\left(y,t\right) = \frac{1}{\lambda\sqrt{t}} \frac{1}{r\left(y,t\right)}$$

and, similarly,

$$\frac{\partial}{\partial x} A\left(y,t,x,0\right) = -\frac{1}{\lambda\sqrt{t}} H_x^{(-1)}\left(x,0\right) = -\frac{1}{\lambda\sqrt{t}} \frac{1}{r\left(x,0\right)},$$

and (41) follows. □

Next, we show how (38) can be used to derive *universal* upper and lower bounds that depend exclusively on the slope of the risk tolerance coefficient $RT(x)$ (cf. (11)). Observe that while (42) holds at terminal time T, the inequalities (43) and (44) hold for *all* $t \in (0,T]$.

Corollary 4.4. *Assume that the absolute risk tolerance coefficient $RT(x)$ satisfies, for $x \geq 0$,*

$$0 < m \leq RT'\left(x\right) \leq n. \tag{42}$$

Then, for $x, y > 0$ and $t \in (0,T]$, the following inequalities hold

$$\mathbb{P}\left(X_t^{*,x} \leq y\right) \leq \Phi\left(\frac{1}{m}\frac{1}{\lambda\sqrt{t}}\log\left(\frac{y}{x}\right) + \lambda\sqrt{t}\left(\frac{n}{2}-1\right)\right) \tag{43}$$

and

$$\mathbb{P}\left(X_t^{*,x} \leq y\right) \geq \Phi\left(\frac{1}{n}\frac{1}{\lambda\sqrt{t}}\log\left(\frac{y}{x}\right) + \lambda\sqrt{t}\left(\frac{m}{2}-1\right)\right). \tag{44}$$

Proof. In Källblad and Zariphopoulou (2014) (see Proposition 16) it was shown that if (42) holds, then this property is inherited by the local absolute risk tolerance function for all $t \in (0,T]$,

$$0 < m \leq r_x\left(x,t\right) \leq n. \tag{45}$$

Using the above inequality and (38) we conclude. □

4.2 The probability of falling below initial wealth

Next, we consider the probability that the optimal wealth drops, at time $t \in (0,T]$, below the initial wealth. The following result follows directly from (38) and Corollary 4.4.

Proposition 4.5. *i) For $t \in (0,T]$, $x > 0$, we have*

$$\mathbb{P}\left(X_t^{*,x} \leq x\right) = \Phi\left(\frac{\lambda}{2\sqrt{t}}\int_0^t r_x\left(x,s\right)ds - \lambda\sqrt{t}\right). \tag{46}$$

ii) If the absolute risk tolerance coefficient satisfies $0 < m \leq RT'(x) \leq n$, then, for $t \in (0,T]$, $x > 0$,

$$\Phi\left(\lambda\sqrt{t}\left(\frac{m}{2}-1\right)\right) \leq \mathbb{P}\left(X_t^{*,x} \leq x\right) \leq \Phi\left(\lambda\sqrt{t}\left(\frac{n}{2}-1\right)\right). \tag{47}$$

In particular,

$$\mathbb{P}\left(X_t^{*,x} \le x\right) \le \frac{1}{2} \qquad \textit{for } n = 2$$

and

$$\mathbb{P}\left(X_t^{*,x} \le x\right) \ge \frac{1}{2} \qquad \textit{for } m = 2.$$

Next, we look at (46) as a function of x and t. As the following proposition shows, it is *exclusively* the convexity or concavity of the risk tolerance coefficient (cf. (11)) that determines the monotonicity of $\mathbb{P}(X_t^{*,x} \le x)$ with respect to the wealth argument.

The time monotonicity is more involved and we do not have, in general, similar results. However, if the slope of the risk tolerance coefficient is bounded from above and below then, for a certain range of these bounds, we can determine whether $\mathbb{P}(X_t^{*,x} \le x)$ increases or decreases with time.

Proposition 4.6. *i) We have*

$$\frac{\partial}{\partial x}\mathbb{P}\left(X_t^{*,x} \le x\right) = \left(\frac{\lambda}{2\sqrt{t}}\int_0^t r_{xx}\left(x,s\right)ds\right)\phi\left(\frac{\lambda}{2\sqrt{t}}\int_0^t r_x\left(x,s\right)ds - \lambda\sqrt{t}\right). \quad (48)$$

If the absolute risk tolerance coefficient $RT\left(x\right)$ is a concave (convex) function of wealth, then, for $t \in (0,T)$, $\mathbb{P}\left(X_t^{*,x} \le x\right)$ is decreasing (increasing) in x.

ii) Moreover,

$$\frac{\partial}{\partial t}\mathbb{P}\left(X_t^{*,x} \le x\right)$$

$$= \frac{\lambda}{2\sqrt{t}}\left(r_x\left(x,t\right) - \frac{1}{2t}\int_0^t r_x\left(x,s\right)ds - 1\right)\phi\left(\frac{\lambda}{2\sqrt{t}}\int_0^t r_x\left(x,s\right)ds - \lambda\sqrt{t}\right). \quad (49)$$

If the absolute risk tolerance coefficient $RT\left(x\right)$ satisfies $0 < m \le RT'\left(x\right) \le n$ (cf. (42)), then, if $m < n < \frac{m}{2}+1$, the cumulative probability $\mathbb{P}\left(X_t^{*,x} \le x\right)$ is decreasing in time, while if $m < \frac{n}{2}+1 < n$, it is increasing in time.

Proof. i) In Källblad and Zariphopoulou (2014) (see Proposition 12) it was shown that if $RT\left(x\right)$ is a concave (convex) function of wealth, then $r\left(x,t\right)$ is also concave (convex), for each $t \in (0,T)$. Using this result, we easily conclude.

ii) Using that the bounds of $RT'\left(x\right)$ yield the same bounds for $r_x\left(x,t\right), t \in (0,T)$, (cf.(45)), we easily deduce that for $(x,t) \in \mathbb{R}_+ \times (0,T)$,

$$m - \frac{n}{2} - 1 \le r_x\left(x,t\right) - \frac{1}{2t}\int_0^t r_x\left(x,s\right)ds - 1 \le n - \frac{m}{2} - 1,$$

and the rest of the proof follows easily. $\square$

4.3 Expectation of functionals of the optimal wealth

We next compute the expectation of a functional of the optimal wealth process at a fixed time, which we then use to derive expressions for its mean and variance.

Proposition 4.7. *Let λ be as in* (2)*, H the solution to* (15) *and* (16)*, and a function $g\colon \mathbb{R}_+ \to \mathbb{R}$ of polynomial growth. Let $G\colon \mathbb{R}\times(0,T]\to\mathbb{R}$ be given by*

$$G(x,t) = g(H(x,t)). \tag{50}$$

Then, for $t\in(0,T]$, the expectation $E(g(X_t^{*,x}))$ is given by the convolution

$$E(g(X_t^{*,x})) = \left(G(\cdot,t) * \xi(\cdot,t)\right)(z)\Big|_{z=H^{(-1)}(x,0)+\lambda^2 t} \tag{51}$$

where $\xi\left(x,t\right)$ is the fundamental solution

$$\xi(x,t) = \frac{1}{\sqrt{2\lambda^2\pi t}}e^{-\frac{x^2}{2\lambda^2 t}}. \tag{52}$$

Proof. Recalling the density function in (39), we obtain

$$E(g(X_t^{*,x})) = \int_0^\infty g\left(y\right) f\left(y,t;x,0\right)dy$$

$$= \int_0^\infty g\left(y\right)\frac{1}{\lambda\sqrt{t}}\phi\left(\frac{H^{(-1)}\left(y,t\right) - H^{(-1)}\left(x,0\right) - \lambda^2 t}{\lambda\sqrt{t}}\right)\frac{dy}{r\left(y,t\right)}.$$

Changing variables to $\eta = H^{(-1)}\left(y,t\right)$, we have $d\eta = H_x^{(-1)}\left(y,t\right)dy = \frac{dy}{r(y,t)}$. Moreover, using that for each $t\in(0,T]$ the function $H\left(x,t\right)$ is of full range (see Proposition 3.1 herein), we easily deduce that

$$E(g(X_t^{*,x})) = \frac{1}{\lambda\sqrt{t}}\int_{-\infty}^\infty g\left(H\left(\eta,t\right)\right)\phi\left(\frac{1}{\lambda\sqrt{t}}\left(\eta - \left(H^{(-1)}\left(x,0\right)+\lambda^2 t\right)\right)\right)d\eta.$$

□

Using the above result for specific choices of $g(x)$, namely, $g(x)=x$ and $g(x)=x^2$, we obtain the following expressions for the mean and variance of the optimal wealth process at a fixed time. A similar expression for the mean was first obtained in Musiela and Zariphopoulou (2010).

Corollary 4.8. *The mean and variance of the optimal wealth process $X_t^{*,x}$ at a fixed time $t\in(0,T]$ are given by*

$$E(X_t^{*,x}) = H\left(H^{(-1)}(x,0)+\lambda^2 t, 0\right), \tag{53}$$

and

$$Var(X_t^{*,x}) = v\left(H^{(-1)}(x,0)+\lambda^2 t, t\right), \tag{54}$$

where the function $v : \mathbb{R} \times (0,T] \to \mathbb{R}_+$ *is given by*

$$v(x,t) = (H^2(\cdot,t) * \xi(\cdot,t))(x) - H^2(x,0), \tag{55}$$

and $\xi(x,t)$ *is as in* (52) *.*

Example 5. Let U be given by the CRRA utility (25). Using (26) and (53), we have

$$E\left(X_t^{*,x}\right) = x \exp\left(\frac{\lambda^2}{\gamma} t\right).$$

Moreover, it is readily computed that

$$\left(H^2(\cdot,t) * \xi(\cdot,t)\right)(x) = \exp\left(\frac{2}{\gamma}x + \frac{2\lambda^2 t}{\gamma^2} + \frac{\lambda^2}{\gamma^2}(T-t)\right).$$

Then, (54) yields

$$\mathrm{Var}(X_t^{*,x}) = x^2 \exp\left(\frac{2\lambda^2 t}{\gamma}\right)\left(\exp\left(\frac{\lambda^2 t}{\gamma^2}\right) - 1\right).$$

5 Quantile, VaR and expected shortfall

Over the past twenty years there has been a shift in measuring financial risk away from the standard deviation of returns and toward alternative measures such as value at risk (VaR) and expected shortfall (ES). In contrast to the standard deviation of returns, which describes both the upside and downside dispersion of the distribution of returns, VaR focuses solely on the potential downside (see, among others, Campbell et al. (2001)). VaR is widely used in the risk management operations of financial institutions as a summary or benchmark measure of the firm's exposure to market risk. We refer the reader to Jorion (1997) and Duffie and Pan (1997), among others, for a comprehensive overview of the use of VaR for financial risk management.

While VaR is used extensively in the financial industry, risk management principles that involve VaR are not so often used in individual portfolio management. Herein, we take a preliminary step in this direction and examine the VaR of the optimal wealth process. We provide a general expression for the investor's VaR at a *fixed time* in terms of the associated harmonic function $H(x,t)$.

We begin with the definition of VaR.

Definition 1. The value at risk (VaR) of the investor's optimal wealth at time $t_0 \in (0,T]$ and with confidence level $\alpha \in (0,1)$, denoted by $\mathrm{VaR}_\alpha = \mathrm{VaR}_\alpha(X_{t_0}^{*,x})$, is the minimum (nonnegative) amount such that the probability of experiencing a loss in wealth at time t_0 greater than VaR_α is at most α. That is,

$$\mathrm{VaR}_\alpha(X_{t_0}^{*,x}) = \inf\left\{y \geq 0 \colon \mathbb{P}\left(x - X_{t_0}^{*,x} \geq y\right) \leq \alpha\right\}. \tag{56}$$

It is evident from (56) that VaR is related to the quantile function of $X_{t_0}^{*,x}$. Next, we compute the quantile function of the optimal wealth process at a fixed time. The following follows directly from (40) (see, also, Musiela and Zariphopoulou (2010)).

Proposition 5.1. *Let the optimal wealth process, $X_t^{*,x}, t \in 0, T]$, be given by* (22)*. Then, at time $t_0 \in [0,T]$, its quantile function, $F^{(-1)}(y, t_0)$, is given by*

$$F^{(-1)}(y, t_0) = H\left(H^{(-1)}(x, 0) + \lambda^2 t_0 + \lambda\sqrt{t_0}\Phi^{(-1)}(y), t_0\right), \tag{57}$$

where $\Phi^{(-1)}$ is the quantile function of the standard normal distribution.

The probability that the investor experiences a loss in his optimal wealth at t_0 is equal to the probability that his optimal wealth at t_0 falls below his initial wealth. This probability was studied in section 4.2 (see (46)). Observe that, for any confidence level α that is greater than or equal to this probability, the investor's VaR at this confidence level is equal to zero. Therefore, the interval for the confidence level α in the definition of VaR can be decomposed into a disjoint union of intervals, $(0, \alpha^*)$ and $[\alpha^*, 1)$, for some maximal effective confidence level α^*, wherein VaR is positive on the former interval and zero on the latter interval. By (46), we have that

$$\alpha^* = \Phi\left(\frac{\lambda}{2\sqrt{t_0}}\int_0^{t_0} r_x(x,s)ds - \lambda\sqrt{t_0}\right), \tag{58}$$

where r is the investor's local absolute risk tolerance function.

We are now ready to compute the VaR of the investor's optimal wealth at a fixed time.

Proposition 5.2. *Let the optimal wealth process, $X_t^{*,x}, t \in [0,T]$, be given by* (22)*. Then, the investor's value at risk, $VaR_\alpha(X_{t_0}^{*,x})$, at time $t_0 \in (0,T]$ and with confidence level $\alpha \in (0, \alpha^*)$, where α^* is as in* (58)*, is given by*

$$VaR_\alpha(X_{t_0}^{*,x}) = x - H\left(H^{(-1)}(x, 0) + \lambda^2 t_0 + \lambda\sqrt{t_0}\Phi^{(-1)}(\alpha), t_0\right), \tag{59}$$

where $\Phi^{(-1)}$ is the quantile function of the standard normal distribution.

Proof. Under the above assumptions, we have

$$\alpha = \mathbb{P}\left(X_{t_0}^{*,x} - x < -\mathrm{VaR}_\alpha\right) = F\left(x - \mathrm{VaR}_\alpha, t_0\right),$$

where $F(y, t_0)$ is the distribution function of $X_{t_0}^{*,x}$. The result then follows from (57). □

5.1 Inferring risk aversion from VaR targets

We provide an example in which we infer the risk preferences of an investor who is an expected utility maximizer in $[0,T]$ but also places a VaR target at a specific time $t_0 \in (0,T]$. Under CRRA utility (cf. (25)) this is equivalent to inferring the coefficient of relative risk aversion γ.

Extracting risk preferences from investment targets has been analyzed in Musiela and Zariphopoulou (2010) and, more recently, in Monin (2014). These papers used the investor's desired distributional data to infer his risk preferences. Specifically, in the former paper the authors used the targeted mean to extract the investor's risk tolerance coefficient, while in the latter paper the investor's marginal utility was recovered from a targeted wealth distribution.

Similarly, we show herein that fixing a VaR target for the investor's optimal wealth at a single fixed time within the investment horizon is sufficient to *uniquely* determine the investor's risk aversion if the investor has CRRA utility.

Proposition 5.3. *Let the investor have CRRA utility* (25) *and suppose the investor targets the VaR for his optimal wealth at time* $t_0 \in (0, T]$ *and with confidence level* $\alpha \in \big(0, \Phi(-\lambda\sqrt{t_0})\big)$ *to be*

$$VaR_\alpha(X_{t_0}^{*,x}) = px, \tag{60}$$

for some proportion $p \in (0, 1)$ *of his initial wealth* x*. Then, the investor's coefficient of relative risk aversion is uniquely given by*

$$\gamma = \frac{\lambda\sqrt{t_0}\Phi^{(-1)}(\alpha) + \lambda^2 t_0 - \sqrt{(\lambda\sqrt{t_0}\Phi^{(-1)}(\alpha) + \lambda^2 t_0)^2 - 2\lambda^2 t_0 \log(1-p)}}{2\log(1-p)}, \tag{61}$$

where $\Phi^{(-1)}$ *is the quantile function of the standard normal distribution.*

Proof. First, observe that (61) is well-defined since $\log(1-p) < 0$ for all $p \in (0, 1)$. Next, for arbitrary $\gamma > 0$, (58) yields that

$$\alpha^* = \alpha^*(\gamma) = \Phi\left(\frac{\lambda\sqrt{t_0}}{2\gamma} - \lambda\sqrt{t_0}\right),$$

which is greater than $\Phi(-\lambda\sqrt{t_0})$ for all $\gamma > 0$. Then, using (59) and (26) yields

$$\mathrm{VaR}_\alpha(X_{t_0}^{*,x}) = x - x\exp\left(\frac{\lambda}{\gamma}\sqrt{t_0}\Phi^{(-1)}(\alpha) + \frac{\lambda^2}{\gamma}\left(1 - \frac{1}{2\gamma}\right)t_0\right).$$

From (60) the above becomes

$$\frac{\lambda}{\gamma}\sqrt{t_0}\Phi^{(-1)}(\alpha) + \frac{\lambda^2}{\gamma}\left(1 - \frac{1}{2\gamma}\right)t_0 = \log(1-p),$$

and, in turn,

$$\gamma^2\log(1-p) - \gamma\left(\lambda\sqrt{t_0}\Phi^{(-1)}(\alpha) + \lambda^2 t_0\right) + \frac{1}{2}\lambda^2 t_0 = 0.$$

Solving this quadratic we deduce that its positive root must be given by (61). □

In Fig. 1 we use (61) to show the implied risk aversion coefficient for an investor who sets a VaR target for his terminal wealth. We do this for various realistic confidence levels α. For each fixed α, it is seen that the lower the investor is willing to risk, in the sense of VaR, the higher is his risk aversion.

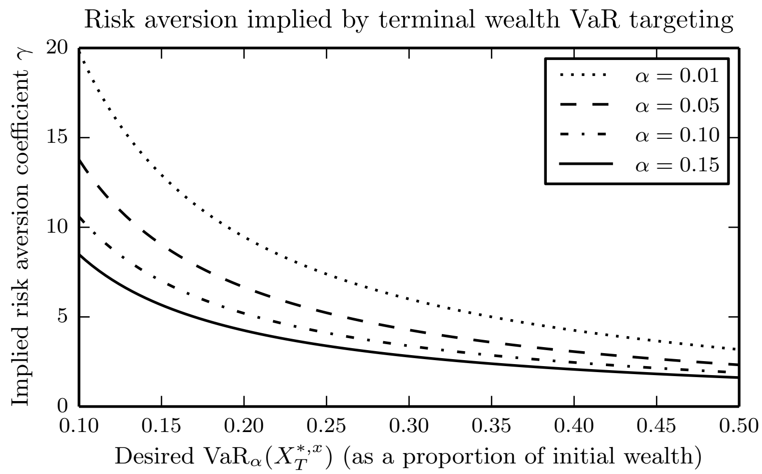

Figure 1: Risk aversion implied by terminal wealth VaR targeting for an investor with CRRA utility. Parameters: $\lambda = 0.15$, $T = 40$.

5.2 Expected shortfall and implied risk preferences

Despite its popularity, VaR has well-known deficiencies as a measure of financial risk. For example, VaR does not provide the investor with an estimate of his expected losses in the event that the VaR level is exceeded. The actual losses when the VaR level is exceeded will be greater than or equal to the VaR level itself. Indeed, losses could be much greater than the VaR level, depending on the shape of the tail of the returns distribution. Another deficiency of VaR is that it generally does not reward diversification, since it is possible for the VaR of a sum of two portfolios to be greater than the sum of the VaRs of the individual portfolios (see Artzner et al. (1997, 1999)). There exist many alternative risk measures to VaR and, among these, a popular one that addresses the above deficiencies is the so-called expected shortfall (ES). ES takes into account the tail of the distribution of losses beyond the VaR level and, unlike VaR, it is a so-called coherent risk measure (see, for example, Acerbi and Tasche (2002)), which implies that it rewards diversification.

Herein, we calculate the ES at a given horizon and confidence level for an investor with CRRA utility. We then discuss the inference of risk preferences for an investor who targets the ES for his optimal wealth at a given time. We begin with the following definition, which can be found in Hult et al. (2012), among others.

Definition 2. The expected shortfall (ES) of the investor's optimal wealth at time $t_0 \in (0, T]$ and with confidence level $\alpha \in (0, 1)$, denoted by $\mathrm{ES}_\alpha(X_{t_0}^{*,x})$, is the expected loss to the investor's wealth conditional on the loss being greater than or equal to $\mathrm{VaR}_\alpha(X_{t_0}^{*,x})$. That is,

$$\mathrm{ES}_\alpha(X_{t_0}^{*,x}) = -E\left(X_{t_0}^{*,x} - x \;\middle|\; X_{t_0}^{*,x} - x \leq -\mathrm{VaR}_\alpha(X_{t_0}^{*,x})\right). \tag{62}$$

We only consider ES_α for $\alpha \in (0, \alpha^*)$, where α^* is given in (58). To see why, recall that $\mathrm{VaR}_\alpha = 0$ for all $\alpha \in [\alpha^*, 1)$, which, by (62), implies that

$$\mathrm{ES}_\alpha(X_{t_0}^{*,x}) = \mathrm{ES}_{\alpha^*}(X_{t_0}^{*,x}) = -E\left(X_{t_0}^{*,x} - x\right), \qquad \alpha \in [\alpha^*, 1).$$

That is, for all $\alpha \in [\alpha^*, 1)$, ES_α is just the (unconditional) expected loss on the optimal wealth.

Given that the optimal wealth $X_{t_0}^{*,x}$ in our model has continuous distribution function (cf. (38)), the expected shortfall at time $t_0 \in (0, T]$ and with confidence level $\alpha \in (0, \alpha^*)$ can be written (see, for example, Hult et al. (2012, Proposition 6.5)) as

$$\mathrm{ES}_\alpha(X_{t_0}^{*,x}) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_z(X_{t_0}^{*,x})dz. \tag{63}$$

The expression (63) is suggestive of the alternative names by which ES is known. These include average VaR (aVaR) and tail conditional expectation (TCE).

Combining the above and Proposition 5.2, we deduce the following.

Proposition 5.4. *Let the optimal wealth process, $X_t^{*,x}, t \in [0, T]$, be given by (22). Then, the investor's expected shortfall, $ES_\alpha(X_{t_0}^{*,x})$, at time $t_0 \in (0, T]$ and with confidence level $\alpha \in (0, \alpha^*)$, where α^* is as in (58), is given by*

$$ES_\alpha(X_{t_0}^{*,x}) = x - \frac{1}{\alpha}\int_0^\alpha H\left(H^{(-1)}(x, 0) + \lambda^2 t + \lambda\sqrt{t}\Phi^{(-1)}(z), t\right) dz, \tag{64}$$

where $\Phi^{(-1)}$ is the quantile function of the standard normal distribution.

We are now ready to calculate the expected shortfall at a given horizon and confidence level for a CRRA investor. We also show that, under a mild additional assumption, the expected shortfall is decreasing in the coefficient of relative risk aversion γ.

Proposition 5.5. *Let the investor have CRRA utility (25). The following assertions hold.*

i) The expected shortfall of the investor's optimal wealth at time $t_0 \in (0, T]$ and with confidence level $\alpha \in (0, \alpha^*)$, where α^* is as in (58), is given by

$$ES_\alpha(X_{t_0}^{*,x}) = x\left(1 - \frac{1}{\alpha}\exp\left(\frac{\lambda^2}{\gamma}t_0\right)\Phi(\Phi^{(-1)}(\alpha) - \frac{\lambda}{\gamma}\sqrt{t_0})\right). \tag{65}$$

ii) If (λ, t_0, α) are such that $\alpha < \Phi(-\lambda\sqrt{t_0})$, then, for fixed (λ, t_0, α), the expected shortfall is a strictly decreasing function of the relative risk aversion coefficient γ.

Proof. i) First, observe that, if $F^{(-1)}(y, t_0)$ is the quantile function of the optimal wealth $X_{t_0}^{*,x}$, it follows that

$$\mathrm{ES}_\alpha(X_{t_0}^{*,x}) = x - E\left(X_{t_0}^{*,x} \mid X_{t_0}^{*,x} < F^{(-1)}(\alpha, t_0)\right),$$

where we have used that $\mathrm{VaR}_\alpha(X_{t_0}^{*,x}) = x - F^{(-1)}(\alpha, t_0)$.

Next, we recall that if a random variable, say Y, satisfies $\log(Y) \sim N(\mu, \sigma^2)$, then, for $\alpha \in (0,1)$,

$$E\left(Y \,|Y < F_Y^{(-1)}(\alpha)\right) = \frac{1}{\alpha} e^{\mu + \frac{1}{2}\sigma^2} \Phi(\Phi^{(-1)}(\alpha) - \sigma),$$

where $F_Y^{(-1)}$ is the quantile function of Y (see, for example, Dhaene et al. (2006, equation (25))).

For CRRA utility, we have (cf. (27)) that

$$\log(X_{t_0}^{*,x}) \sim N\left(\log(x) + \frac{\lambda^2}{\gamma}\left(1 - \frac{1}{2\gamma}\right) t_0, \frac{\lambda^2}{\gamma^2} t_0\right),$$

and we easily conclude.

ii) We first recall that for the standard normal distribution function Φ and its density function ϕ, we have that

$$1 - \Phi(z) \leq \frac{\phi(z)}{z},\ z > 0 \quad \text{ and } \quad \Phi(z) \leq -\frac{\phi(z)}{z},\ z < 0.$$

If $\Phi^{(-1)}(\alpha) < -\lambda\sqrt{t_0}$, then, for $z = \Phi^{(-1)}(\alpha) - \frac{\lambda}{\gamma}\sqrt{t_0} < 0$,

$$\begin{aligned}
\frac{\partial \mathrm{ES}_\alpha}{\partial \gamma} &= -\frac{x}{\alpha}\frac{\partial}{\partial \gamma}\left(\exp\left(\frac{\lambda^2}{\gamma} t_0\right)\Phi(\Phi^{(-1)}(\alpha) - \frac{\lambda}{\gamma}\sqrt{t_0})\right) \\
&= \frac{x\lambda}{\alpha\gamma^2}\sqrt{t_0}\exp\left(\frac{\lambda^2}{\gamma} t_0\right)\left(\lambda\sqrt{t_0}\Phi(z) - \phi(z)\right) \\
&\leq \frac{x\lambda}{\alpha\gamma^2}\sqrt{t_0}\exp\left(\frac{\lambda^2}{\gamma} t_0\right)\left(-\lambda\sqrt{t_0}\frac{\phi(z)}{z} - \phi(z)\right) \\
&= \frac{x\lambda}{\alpha\gamma^2}\sqrt{t_0}\exp\left(\frac{\lambda^2}{\gamma} t_0\right)\phi(z)\left(\frac{\lambda\sqrt{t_0}}{\frac{\lambda}{\gamma}\sqrt{t_0} - \Phi^{(-1)}(\alpha)} - 1\right) \\
&< \frac{x\lambda}{\alpha\gamma^2}\sqrt{t_0}\exp\left(\frac{\lambda^2}{\gamma} t_0\right)\phi(z)\left(-\frac{\lambda\sqrt{t_0}}{\Phi^{(-1)}(\alpha)} - 1\right) < 0.
\end{aligned}$$

□

In Fig. 2 we use (65) to depict the investor's expected shortfall over $[0, T]$ as a function of the coefficient of relative risk aversion γ for various levels of confidence α. Figure 3 shows the inverse dependence, i.e. the investor's relative risk aversion as a function of his desired expected shortfall. This can be done since the parameter values for (λ, T, α) in the numerical computation satisfy $\alpha < \Phi(-\lambda\sqrt{T})$ which, by Proposition 5.5(ii), is a sufficient condition under which the implied coefficient of relative risk aversion γ can be found by numerically inverting (61). Similarly to Fig. 1, we see that, for each fixed confidence level α, a lower desired expected shortfall implies a higher inferred risk aversion coefficient γ.

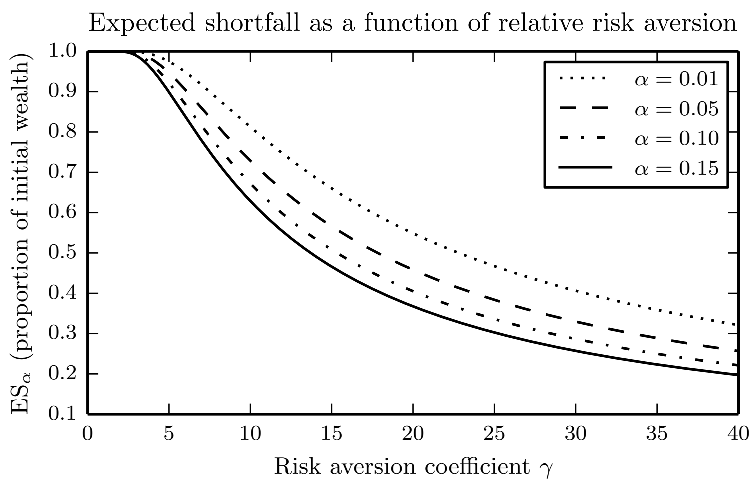

Figure 2: Expected shortfall for an investor with CRRA utility as a function of risk aversion. Parameters: $\lambda = 0.15$, $T = 40$.

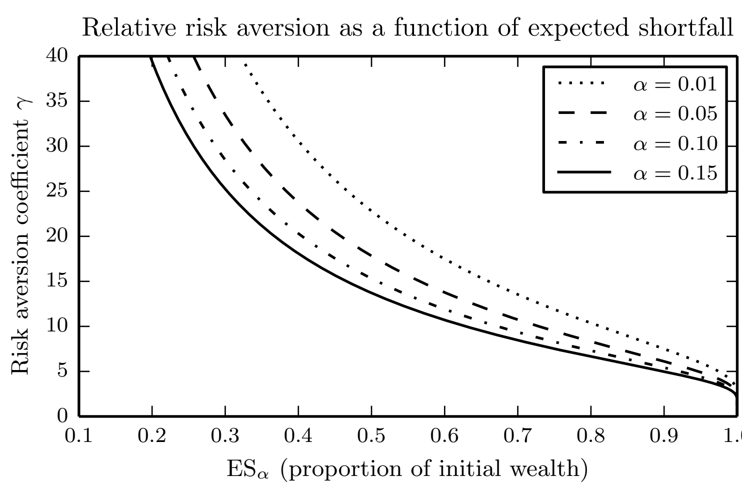

Figure 3: Risk aversion for an investor with CRRA utility as a function of expected shortfall. Parameters: $\lambda = 0.15$, $T = 40$.

6 Portfolio greeks for utility-based investment

Sensitivity analysis is the process of evaluating incremental impacts on value of changes in underlying individual variables on which the value depends. The most common application of sensitivity analysis in mathematical finance is the use of so-called option greeks, e.g. “delta” and “gamma,” for hedging derivative exposures. Herein, we introduce *portfolio greeks*, drawing an analogy to the well-known option greeks. Portfolio greeks for utility-based investment are then naturally defined as the sensitivities of the investor’s optimal wealth with respect to the various underlying parameters on which it depends. To the best of our knowledge, these sensitivities have not been considered before in the literature.

Both option greeks and portfolio greeks find applications in risk management. In the case of options, the greeks are typically used to hedge risk in derivative exposures. Institutions transact in derivatives for the fees they collect, and they will often try to hedge the risk in the exposure rather than retain it. While risk reduction is the typical application of option greeks, this is not necessarily the case in optimal investment, which is based on exploiting risk in accordance with the investor’s preferences. Nonetheless, risk management for individual investors is an essential, though perhaps overlooked, part of the investment process. Portfolio greeks can be used to estimate the sensitivities of the investor’s wealth to underlying market parameters and then to manage risk through sensitivity analysis and stress testing.

In contrast to the options greeks delta and gamma, in which the price level of the stock is the relevant state variable, the appropriate state variable for portfolio greeks seems to be the return on the stock. To see this, consider that for a standard European call or put option, it is sufficient to know the stock price level to determine the current value of the option. To determine an investor’s wealth in investment management, however, the stock price level is insufficient. Instead, one must know not only the current stock price level but also all of the stock price levels at which the investor transacted in the stock in the past. In optimal investment, therefore, one must know the *cumulative return* on the stock over the investment period to determine the investor’s wealth at a given time. Moreover, given that we work herein with discounted wealth (in which the riskless asset is the numéraire), the relevant state variable in our model is the cumulative *excess* return of the stock over that of the riskless asset.

We are now ready to define portfolio greeks for utility-based investment. Let the stock price S_t, $t \in [0,T]$, be as in (1). Define the stock’s *cumulative (continuously compounded) excess return* process, R_t, $t \in [0,T]$, as

$$R_t = \log\left(\frac{S_t}{S_0}\right) - rt, \tag{66}$$

and the *mean excess return* of the stock as $\widehat{\mu} = \mu - r$. Then, (1) yields

$$dR_t = (\widehat{\mu} - \frac{1}{2}\sigma^2)dt + \sigma dW_t, \tag{67}$$

with $R_0 = 1$. The optimal wealth process $X_t^{*,x}$ can be written as a *harmonic function of the process* R_t, namely,

$$X_t^{*,x} = C\left(R_t, t; x\right) \tag{68}$$

with $C : \mathbb{R}\times [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$C\left(R, t; x\right) = H\left(H^{(-1)}(x,0) + \frac{1}{2}\widehat{\mu}t + \frac{\lambda}{\sigma}R, t\right). \tag{69}$$

In particular,

$$C(R,T;x) = I\left(\exp\left(-(H^{(-1)}(x,0) + \frac{1}{2}\widehat{\mu}T + \frac{\lambda}{\sigma}R)\right)\right), \tag{70}$$

with I as in (5).

Because of representation (68), we will occasionally refer to $C(R,t;x)$ as the *wealth function*.

Recall that in the log-normal market model considered herein, the price at a time before maturity of an option written on the stock is given by a deterministic function of time and the stock price, where the function satisfies the Black-Scholes-Merton partial differential equation with terminal condition given in terms of the option payoff. Analogously, (68) and (69) show that the investor's optimal wealth at a time within the investment horizon is given by a *deterministic function* of time and the stock's cumulative excess return, where the function now satisfies a terminal condition (see (70)) given in terms of the investor's risk preferences.

Next, we introduce two auxiliary functions $\Delta, \Gamma\colon \mathbb{R}\times [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\Delta\left(R, t; x\right) = H_x\left(H^{(-1)}(x,0) + \frac{1}{2}\widehat{\mu}t + \frac{\lambda}{\sigma}R, t\right) \tag{71}$$

and

$$\Gamma\left(R, t; x\right) = H_{xx}\left(H^{(-1)}(x,0) + \frac{1}{2}\widehat{\mu}t + \frac{\lambda}{\sigma}R, t\right), \tag{72}$$

and calculate the sensitivities of the wealth function $C\left(R,t;x\right)$ in terms of variable R.

Lemma 6.1. *i) The sensitivity of $C(R,t;x)$ with respect to R is given by*

$$\frac{\partial C\left(R,t;x\right)}{\partial R} = \frac{\lambda}{\sigma}\Delta\left(R,t;x\right), \tag{73}$$

while its convexity by

$$\frac{\partial^2 C\left(R,t;x\right)}{\partial R^2} = \left(\frac{\lambda}{\sigma}\right)^2 \Gamma\left(R,t;x\right). \tag{74}$$

More generally,

$$\frac{\partial^n C\left(R,t;x\right)}{\partial R^n} = \left(\frac{\lambda}{\sigma}\right)^n \frac{\partial^n}{\partial x^n} H\left(H^{(-1)}(x,0) + \frac{1}{2}\widehat{\mu}t + \frac{\lambda}{\sigma}R, t\right). \tag{75}$$

ii) For $n = 1, ...,$, and $I(x)$ as in (5)*, if $\frac{\partial^n}{\partial x^n} I(e^{-x}) \gtrless 0$, then, for $(R, t; x) \in \mathbb{R} \times [0, T) \times \mathbb{R}_+$,*

$$\frac{\partial^n C(R, t; x)}{\partial R^n} \gtrless 0. \tag{76}$$

Proof. Part *i)* follows by direct differentiation.

For part *ii)* we recall that all partials $\frac{\partial^n}{\partial x^n} H(x, t)$, $n = 1, ...$, solve the heat equation (15) with terminal condition $\frac{\partial^n}{\partial x^n} H(x, T) = \frac{\partial^n}{\partial x^n} I(e^{-x})$. Using (6) we deduce that the appropriate conditions for the application of the comparison principle hold, and we easily conclude. □

In the options literature, the sensitivities of an option's value with respect to the parameters of the underlying model are widely studied, usually in a hedging context. An option's delta, for example, represents the incremental change in the value of the option with respect to the incremental change in the stock price, while an option's gamma represents the incremental change in the option's delta with respect to an incremental change in the stock price. These sensitivities, denoted by $\frac{\partial V_t}{\partial S_t}$ and $\frac{\partial^2 V_t}{\partial S_t^2}$, respectively, where V_t is the value of the option, are formally defined constructions that are found by differentiating a deterministic pricing function that gives the option's price in terms of the model's parameters and then evaluating the result at the stock's price. Herein, we take a similar approach to computing sensitivities of the optimal wealth process. Namely, we compute first- and second-order sensitivities of the optimal wealth with respect to the stock's cumulative excess return, which we formally denote by $\frac{\partial X_t^{*,x}}{\partial R_t}$ and $\frac{\partial^2 X_t^{*,x}}{\partial R_t^2}$.

We start with the following lemma.

Lemma 6.2. *Let R_t, $t \in [0, T]$, be as in* (66)*. Then,*

$$\Delta(R_t, t; x) = r\left(X_t^{*,x}, t\right) \tag{77}$$

and

$$\Gamma(R_t, t; x) = r\left(X_t^{*,x}, t\right) r_x\left(X_t^{*,x}, t\right), \tag{78}$$

where r is the local absolute risk tolerance function (cf.(12)*) and $X_t^{*,x}, t \in [0, T]$, the optimal wealth process.*

Proof. Equality (77) follows from (23). To show (78), we first observe that (17) yields

$$\begin{aligned} H_{xx}\left(H^{(-1)}(x,t), t\right) &= \frac{H_{xx}\left(H^{(-1)}(x,t), t\right)}{H_x(H^{(-1)}(x,t), t)} H_x(H^{(-1)}(x,t), t) \\ &= r_x(x,t)\, r(x,t), \end{aligned} \tag{79}$$

and we easily conclude. □

Proposition 6.3. *Let $X_t^{*,x}$ and $\pi_t^{*,x}$ be the investor's optimal wealth and portfolio processes given, respectively, by* (22) *and* (23)*, and $r(x, t)$ be the local absolute risk tolerance function. The following assertions hold.*

i) The sensitivity of $X_t^{*,x}$ with respect to R_t is given by

$$\frac{\partial X_t^{*,x}}{\partial R_t} := \left.\frac{\partial C\left(R,t;x\right)}{\partial R}\right|_{R=R_t} = \pi_t^{*,x}. \tag{80}$$

ii) The convexity of $X_t^{*,x}$ with respect to R_t is given by

$$\frac{\partial^2 X_t^{*,x}}{\partial R_t^2} := \left.\frac{\partial^2 C\left(R,t;x\right)}{\partial R^2}\right|_{R=R_t} = \frac{\lambda}{\sigma} r_x\left(X_t^{*,x},t\right)\pi_t^{*,x} \tag{81}$$

$$= \left(\frac{\lambda}{\sigma}\right)^2 \left(\frac{1}{2}\left.\frac{\partial}{\partial x} r^2(x,t)\right|_{x=X_t^{*,x}}\right). \tag{82}$$

iii) The change in the investor's wealth associated to a change in the continuously excess return on the stock can be approximated by

$$\Delta X_t^{*,x} \approx \frac{\lambda}{\sigma} r(X_t^{*,x},t)\Delta R_t + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 \left(\frac{1}{2}\left.\frac{\partial}{\partial x} r^2(x,t)\right|_{x=X_t^{*,x}}\right)(\Delta R_t)^2. \tag{83}$$

In Fig. 4 we depict the wealth function C as a function of R for an investor with CRRA utility (25). That is, the function H used to calculate (69) is given by (26). The figure illustrates how the slope and convexity effects of the cumulative excess return on the wealth function depend on the risk preferences of the investor. We observe that, the more risk averse an investor is, the less are the slope and convexity effects on his wealth function.

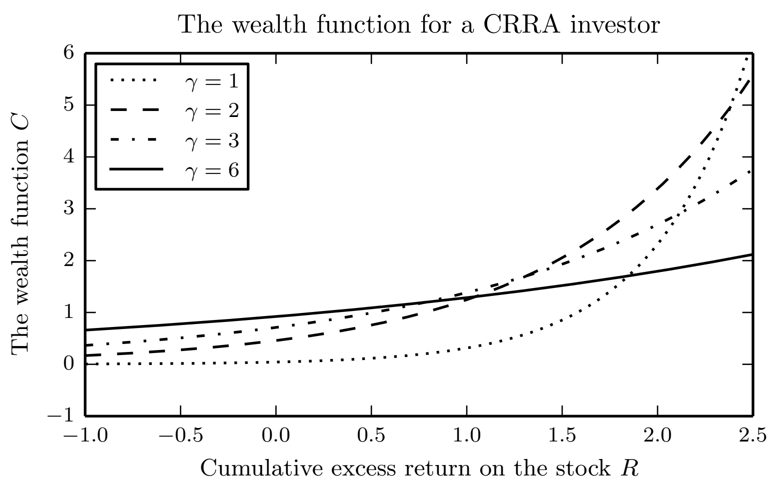

Figure 4: The wealth function $C(R,t;x)$ for an investor with CRRA utility as a function of the cumulative excess return on the stock, for various levels of risk aversion γ. Parameters: $\widehat{\mu} = 0.08, \sigma = 0.20, \lambda = 0.40, T = 40, x = 1$.

We continue with the sensitivities of the wealth function $C(R,t;x)$ with respect to time t and the market parameters σ and $\widehat{\mu}$.

Lemma 6.4. *We have*

$$\frac{\partial C\left(R,t;x\right)}{\partial t}=\frac{1}{2}\widehat{\mu}\Delta\left(R,t;x\right)-\frac{1}{2}\lambda^{2}\Gamma\left(R,t;x\right). \tag{84}$$

Moreover,

$$\frac{\partial C\left(R,t;x\right)}{\partial \sigma}$$
$$=\left(-\frac{2\lambda}{\sigma^{2}}R+\frac{\lambda^{2}T}{\sigma}r_{x}\left(x,0\right)\right)\Delta\left(R,t;x\right)-\frac{\lambda^{2}(T-t)}{\sigma}\Gamma\left(R,t;x\right) \tag{85}$$

and

$$\frac{\partial C\left(R,t;x\right)}{\partial \widehat{\mu}}$$
$$=\left(\frac{1}{2}t-\frac{\lambda T}{\sigma}r_{x}(x,0)+\frac{1}{\sigma^{2}}R\right)\Delta\left(R,t;x\right)+\frac{\lambda(T-t)}{\sigma}\Gamma\left(R,t;x\right). \tag{86}$$

Proof. Assertion (84) follows easily from (15). To show (85), we have

$$\frac{\partial C\left(R,t;x\right)}{\partial \sigma}=H_{\sigma}\left(H^{(-1)}(x,0;\sigma)+\frac{1}{2}\widehat{\mu}t+\frac{\lambda}{\sigma}R,t;\sigma\right)$$
$$+\Delta\left(R,t;x,\sigma\right)\frac{\partial}{\partial\sigma}\left(H^{(-1)}(x,0;\sigma)+\frac{1}{2}\widehat{\mu}t+\frac{\lambda}{\sigma}R\right).$$

Recall (cf. (15)) that $H_{t}+\frac{1}{2}\left(\frac{\widehat{\mu}}{\sigma}\right)^{2}H_{xx}=0$. Define $h:\mathbb{R}_{+}\to\mathbb{R}_{+}$ as

$$h(z)=\frac{1}{\sqrt{2\pi\widehat{\mu}^{2}z}}\int_{-\infty}^{\infty}I(e^{-y})\exp\left(-\frac{(x-y)^{2}}{2\widehat{\mu}^{2}z}\right)dy,$$

and observe that $H(x,t;\sigma)=h\left(\frac{T-t}{\sigma^{2}}\right)$. Then,

$$\frac{\partial H(x,t;\sigma)}{\partial t}=-\frac{1}{\sigma^{2}}h'\left(\frac{T-t}{\sigma^{2}}\right),\qquad \frac{\partial H(x,t;\sigma)}{\partial \sigma}=-2\frac{(T-t)}{\sigma^{3}}h'\left(\frac{T-t}{\sigma^{2}}\right),$$

from which we deduce that

$$\frac{\partial H(x,t;\sigma)}{\partial \sigma}=\frac{2(T-t)}{\sigma}\frac{\partial H(x,t;\sigma)}{\partial t}=-\frac{\lambda^{2}(T-t)}{\sigma}\frac{\partial^{2}H(x,t;\sigma)}{\partial x^{2}}.$$

Next, from $H\left(H^{(-1)}(x,t;\sigma),t;\sigma\right)=x$, we have

$$H_{\sigma}^{(-1)}(x,t;\sigma)=-\frac{H_{\sigma}(H^{(-1)}(x,t;\sigma),t;\sigma)}{H_{x}(H^{(-1)}(x,t;\sigma),t;\sigma)}$$
$$=\frac{\lambda^{2}(T-t)}{\sigma}\frac{H_{xx}(H^{(-1)}(x,t;\sigma),t;\sigma)}{H_{x}(H^{(-1)}(x,t;\sigma),t;\sigma)}=\frac{\lambda^{2}(T-t)}{\sigma}r_{x}(x,t;\sigma).$$

Combining the above we easily obtain (85).

To show (86), we work similarly. To this end, let $\widehat{h}:\mathbb{R}_+\to\mathbb{R}_+$ be given by

$$\widehat{h}(z)=\frac{1}{\sqrt{2\pi z/\sigma^2}}\int_{-\infty}^{\infty}I(e^{-y})\exp\left(-\frac{(x-y)^2}{2z/\sigma^2}\right)dy,$$

and observe that $H(x,t;\widehat{\mu})=\widehat{h}(\widehat{\mu}^2(T-t))$. Then,

$$\frac{\partial H(x,t;\widehat{\mu})}{\partial t}=-\widehat{\mu}^2\widehat{h}'(\widehat{\mu}^2(T-t)),\qquad \frac{\partial H(x,t;\widehat{\mu})}{\partial\widehat{\mu}}=2\widehat{\mu}(T-t)\widehat{h}'(\widehat{\mu}^2(T-t)),$$

from which we deduce that

$$\frac{\partial H(x,t;\widehat{\mu})}{\partial\widehat{\mu}}=-\frac{2(T-t)}{\widehat{\mu}}\frac{\partial H(x,t;\widehat{\mu})}{\partial t}=\frac{\lambda^2(T-t)}{\widehat{\mu}}\frac{\partial^2 H(x,t;\widehat{\mu})}{\partial x^2}.$$

We easily deduce that

$$H_{\widehat{\mu}}^{(-1)}(x,t;\widehat{\mu})=-\frac{\lambda^2(T-t)}{\widehat{\mu}}r_x(x,t;\widehat{\mu}).$$

Therefore

$$\begin{aligned}\frac{\partial C\left(R,t;x\right)}{\partial\widehat{\mu}}&=H_{\widehat{\mu}}\left(H^{(-1)}(x,0;\widehat{\mu})+\frac{1}{2}\widehat{\mu}t+\frac{\widehat{\mu}}{\sigma^2}R,t;\widehat{\mu}\right)\\&+\Delta\left(R,t;x\right)\frac{\partial}{\partial\widehat{\mu}}\left(H^{(-1)}(x,0;\widehat{\mu})+\frac{1}{2}\widehat{\mu}t+\frac{\widehat{\mu}}{\sigma^2}R\right).\\&=\Delta\left(R,t;x\right)\left(\frac{1}{2}t-\frac{\lambda^2T}{\widehat{\mu}}r_x(x,0)+\frac{1}{\sigma^2}R\right)+\frac{\lambda^2(T-t)}{\widehat{\mu}}\Gamma\left(R,t;x\right).\end{aligned}$$

□

Using the above and Lemma 6.2 we readily obtain the sensitivities of $X_t^{*,x}$ with respect to t,σ and $\widehat{\mu}$.

Proposition 6.5. *Let $X_t^{*,x}$ and $\pi_t^{*,x}$ be the investor's optimal wealth and portfolio processes given, respectively, by (22) and (23), and $r\left(x,t\right)$ be the local absolute risk tolerance function. The following assertions hold.*

i) The sensitivity of $X_t^{*,x}$ with respect to t is given by

$$\begin{aligned}\frac{\partial X_t^{*,x}}{\partial t}&:=\left.\frac{\partial C\left(R,t;x\right)}{\partial t}\right|_{R=R_t}\\&=\frac{1}{2}\widehat{\mu}r(X_t^{*,x},t)-\frac{1}{2}\lambda^2\left(\frac{1}{2}\left.\frac{\partial}{\partial x}r^2(x,t)\right|_{x=X_t^{*,x}}\right)\\&=\left(\frac{1}{2}\sigma^2-\frac{1}{2}\widehat{\mu}r_x(X_t^{*,x},t)\right)\pi_t^{*,x}.\end{aligned}\tag{87}$$

ii) The sensitivity of $X_t^{*,x}$ with respect to σ is given by

$$\frac{\partial X_t^{*,x}}{\partial \sigma} := \left.\frac{\partial C\,(R,t;x)}{\partial \sigma}\right|_{R=R_t}$$

$$= \left(\frac{\lambda^2 T}{\sigma} r_x(x,0) - \frac{\lambda^2 (T-t)}{\sigma} r_x(X_t^{*,x},t) - \frac{2\lambda}{\sigma^2} R_t\right) r(X_t^{*,x},t)$$

$$= \left(\lambda T r_x(x,0) - \lambda (T-t) r_x(X_t^{*,x},t) - \frac{2}{\sigma} R_t\right) \pi_t^{*,x}. \tag{88}$$

iii) The sensitivity of $X_t^{*,x}$ with respect to $\widehat{\mu}$ is given by

$$\frac{\partial X_t^{*,x}}{\partial \widehat{\mu}} := \left.\frac{\partial C\,(R,t;x)}{\partial \widehat{\mu}}\right|_{R=R_t}$$

$$= \left(\frac{t}{2} - \frac{\lambda T}{\sigma} r_x(x,0) + \frac{\lambda (T-t)}{\sigma} r_x(X_t^{*,x},t) + \frac{1}{\sigma^2} R_t\right) r(X_t^{*},t)$$

$$= \left((T-t) r_x(X_t^{*,x},t) - T r_x(x,0) + \frac{\sigma}{2\lambda} t + \frac{1}{\widehat{\mu}} R_t\right) \pi_t^{*,x}. \tag{89}$$

In Figs. 5, 6 and 7, we represent the sensitivities of the function $C(R,t;x)$ with respect to t, σ, and $\widehat{\mu}$, respectively, for a CRRA investor with utility (25). The function H in (69) is therefore given by (26).

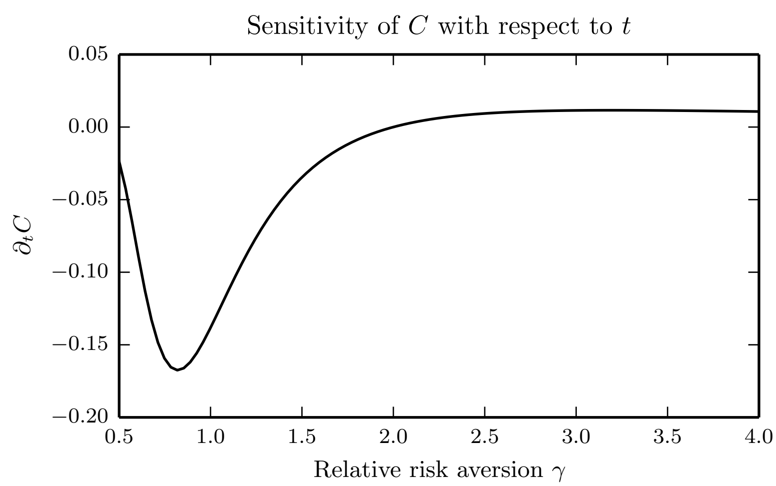

Figure 5: Sensitivity of the wealth function $C(R,t;x)$ with respect to t for an investor with CRRA utility. Parameters: $\widehat{\mu} = 0.08, \sigma = 0.20, \lambda = 0.40, R = 1.8, t = 30, T = 40, x = 1$.

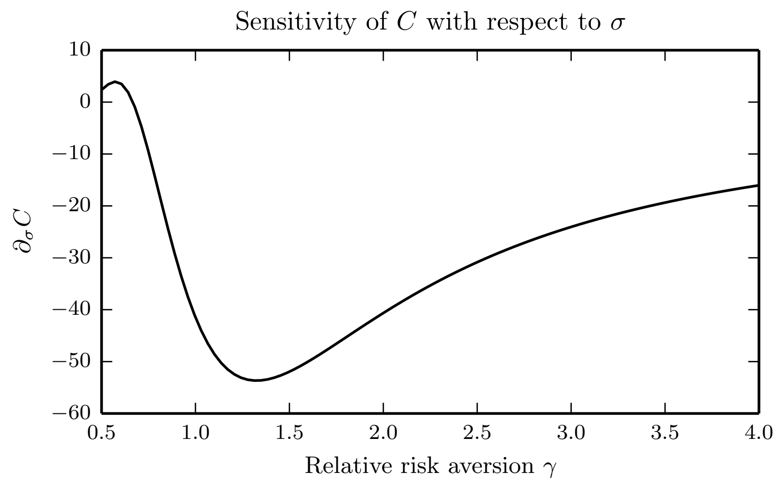

Figure 6: Sensitivity of the wealth function $C(R, t; x)$ with respect to σ for an investor with CRRA utility. Parameters: $\widehat{\mu} = 0.08, \sigma = 0.20, \lambda = 0.40, R = 1.8, t = 30, T = 40, x = 1$.

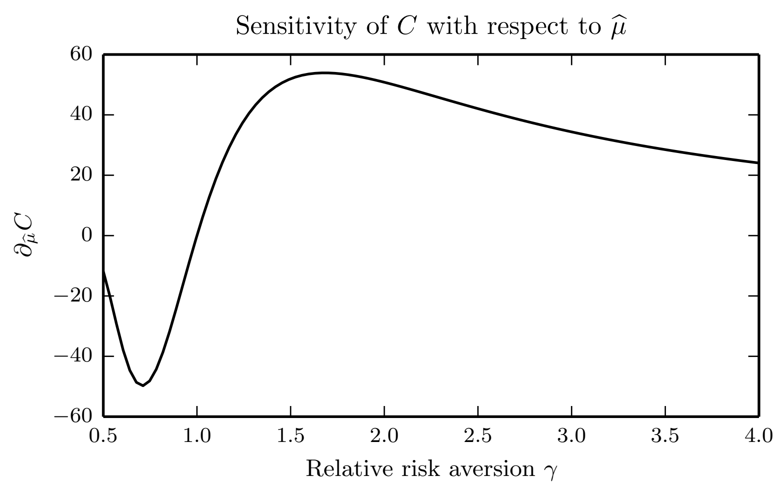

Figure 7: Sensitivity of the wealth function $C(R, t; x)$ with respect to $\widehat{\mu}$ for for an investor with CRRA utility. Parameters: $\widehat{\mu} = 0.08, \sigma = 0.20, \lambda = 0.40, R = 1.8, t = 30, T = 40, x = 1$.

Example 6. Let the investor have CRRA utility (25) as in Example 1. Recall that $\pi_t^{*,x} = \frac{\lambda}{\sigma\gamma} X_t^{*,x}$. By using (17) and (26), it is easily seen that $r(x,t) = \frac{1}{\gamma}x$. Then (80) and (81) yield that

$$\frac{\partial X_t^{*,x}}{\partial R_t} = \frac{\lambda}{\sigma\gamma} X_t^{*,x}, \qquad \text{and} \qquad \frac{\partial^2 X_t^{*,x}}{\partial R_t^2} = \left(\frac{\lambda}{\sigma\gamma}\right)^2 X_t^{*,x}.$$

Furthermore, (87), (88) and (89), respectively, yield that

$$\frac{\partial X_t^{*,x}}{\partial t} = \frac{1}{2}\left(\frac{\widehat{\mu}}{\gamma} - \frac{\lambda^2}{\gamma^2}\right) X_t^{*,x},$$

$$\frac{\partial X_t^{*,x}}{\partial \sigma} = \left(\frac{\lambda^2}{\gamma^2\sigma} t - \frac{2\lambda}{\gamma\sigma^2} R_t\right) X_t^{*,x}$$

and

$$\frac{\partial X_t^{*,x}}{\partial \widehat{\mu}} = \left(\left(\frac{1}{2\gamma} - \frac{\lambda}{\gamma^2\sigma}\right) t + \frac{1}{\gamma\sigma^2} R_t\right) X_t^{*,x}.$$

We make the following observations. First, the investor's optimal wealth is increasing and convex in the stock's cumulative excess return R_t. In addition, the investor's optimal wealth is increasing in time, i.e. $\frac{\partial X_t^{*,x}}{\partial t} > 0$, if and only if the investor's risk aversion coefficient satisfies $\gamma > \frac{\lambda}{\sigma}$.

To discuss monotonicity properties for the other sensitivities, it is convenient to first define the process A_t, $t \in (0,T]$, as the average excess return on the stock, given by $A_t := \frac{1}{t} R_t$. Then, standard but tedious calculations yield the following results. The investor's optimal wealth is increasing in $\widehat{\mu}$, i.e. $\frac{\partial X_t^{*,x}}{\partial \widehat{\mu}} > 0$, if and only if $\gamma\left(A_t + \frac{1}{2}\sigma^2\right) > \widehat{\mu}$. Finally, the investor's optimal wealth is increasing in the stock volatility, i.e. $\frac{\partial X_t^{*,x}}{\partial \sigma} > 0$, if and only if $\gamma A_t < \frac{1}{2}\widehat{\mu}$.

6.1 Beta: That other greek

In Propositions 6.3 and 6.5 we calculated the sensitivities of the level of the optimal wealth $X_t^{*,x}$ with respect to various quantities. Next, we focus on the sensitivities of the *cumulative excess return* of the optimal wealth, rather than its level. We show how these sensitivities relate to the *beta* of the investor's portfolio.

Let $X_t^{*,x}$, $t \in [0,T]$, be the investor's wealth process. We define the optimal wealth's *cumulative (continuously compounded) excess return* process, denoted by R_t^X, $t \in [0,T]$, as

$$R_t^X = \log\left(\frac{X_t^{*,x}}{X_0^{*,x}}\right) = B(R_t, t; x), \tag{90}$$

where R_t is as in (66) and $B\colon \mathbb{R} \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$B(R,t;x) = \log C(R,t;x) - \log x, \tag{91}$$

with $C(R,t;x)$ is as in (69). We will occasionally call $B(R,t;x)$ the *wealth return function.*

Next, we calculate the sensitivities of $B(R,t;x)$ in terms of the variable R. The following lemma follows by direct differentiation.

Lemma 6.6. *Let the functions C, Δ, and Γ be given by (69), (71) and (72), respectively. Then, the sensitivity of $B(R,t;x)$ with respect to R is given by*

$$\frac{\partial B(R,t;x)}{\partial R} = \frac{\lambda}{\sigma}\frac{\Delta(R,t;x)}{C(R,t;x)}, \tag{92}$$

while its convexity is given by

$$\frac{\partial^2 B(R,t;x)}{\partial R^2} = \left(\frac{\lambda}{\sigma}\right)^2\left(\frac{\Gamma(R,t;x)}{C(R,t;x)} - \frac{\Delta^2(R,t;x)}{C^2(R,t;x)}\right). \tag{93}$$

In Propositions 6.3 and 6.5 we showed that the sensitivities of the level of the optimal wealth can be expressed in terms of the optimal portfolio process. We next show that, when considering the sensitivities of the cumulative excess return on the optimal wealth, it is appropriate instead to work with the *optimal portfolio weight process*, that is, the optimal proportion of the investor's wealth that is invested in the stock.

Proposition 6.7. *Let $X_t^{*,x}$ and $\pi_t^{*,x}$ be the investor's optimal wealth and portfolio processes given, respectively, by (22) and (23), and $r(x,t)$ be the local absolute risk tolerance function. Define the optimal portfolio weight process, $\tilde{\pi}_t^{*,x}$, as*

$$\tilde{\pi}_t^{*,x} = \frac{\pi_t^{*,x}}{X_t^{*,x}}. \tag{94}$$

The following assertions hold.

i) The sensitivity of R_t^X with respect to R_t is given by

$$\frac{\partial R_t^X}{\partial R_t} := \left.\frac{\partial B(R,t;x)}{\partial R}\right|_{R=R_t} = \tilde{\pi}_t^{*,x}. \tag{95}$$

ii) The convexity of R_t^X with respect to R_t is given by

$$\frac{\partial^2 R_t^X}{\partial R_t^2} := \left.\frac{\partial^2 B(R,t;x)}{\partial R^2}\right|_{R=R_t} = \left(\frac{\lambda}{\sigma} r_x(X_t^{*,x},t) - \tilde{\pi}_t^{*,x}\right)\tilde{\pi}_t^{*,x}. \tag{96}$$

iii) The change in the cumulative excess return on the investor's optimal wealth associated to a change in the cumulative excess return on the stock can be approximated by

$$\Delta R_t^X \approx \frac{\lambda}{\sigma}\frac{r(X_t^{*,x},t)}{X_t^{*,x}}\Delta R_t + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2\left(r_x(X_t^{*,x},t) - \frac{r(X_t^{*,x},t)}{X_t^{*,x}}\right)\frac{r(X_t^{*,x},t)}{X_t^{*,x}}\left(\Delta R_t\right)^2.$$

We now relate the above sensitivities to the portfolio's beta, where beta is in the sense of the Capital Asset Pricing Model (CAPM) (see Sharpe (1964); Lintner (1965); Mossin (1966)). The CAPM describes the relationship that one should expect between risk and return for individual assets and portfolios. Under many simplifying assumptions, the theory asserts that the expected return on an asset can be computed as the linear combination of the return on a risk-free asset and the expected excess return on the market portfolio, which is the portfolio of all marketable assets weighted in proportion to their relative market values. The sensitivity of an asset's expected excess return to the market's expected excess return is referred to as the *asset's beta*, and is in practice estimated as a single-factor model by regressing asset excess returns on those of a representative market index. Calculating a *portfolio's beta* then involves taking a weighted average of the betas of the constituent stocks within the portfolio, where the weight for a given stock's beta is given by the proportion of the investor's total wealth invested in that stock. Ultimately, then, the portfolio's beta describes the sensitivity of the excess return on the portfolio with respect to the excess return on the market.

In our model, there is one stock that represents the market index. The investor's portfolio's beta is therefore the optimal weight process, $\tilde{\pi}_t^{*,x}$, which by the above, describes the sensitivity of the excess return of the optimal wealth to the excess return of the market. Notice that this is precisely what assertion (95) says.

We conclude by computing sensitivities of R_t^X with respect to the rest of the market parameters. The sensitivities of the process R_t^X with respect to t, σ, and $\widehat{\mu}$ have similar representations to those in Propositions 6.3 and 6.5. The difference is that the optimal portfolio process $\pi_t^{*,x}$ is replaced with the optimal portfolio weight process $\tilde{\pi}_t^{*,x}$.

Proposition 6.8. *Let $X_t^{*,x}$ and $\pi_t^{*,x}$ be the investor's optimal wealth and portfolio processes given, respectively, by* (22) *and* (23)*, and $r(x,t)$ be the local absolute risk tolerance function. Let the optimal portfolio weight process, $\tilde{\pi}_t^{*,x}$, be as in* (94)*. The following assertions hold.*

i) The sensitivity of R_t^X with respect to t is given by

$$\frac{\partial R_t^X}{\partial t} := \left.\frac{\partial B\left(R,t;x\right)}{\partial t}\right|_{R=R_t}$$

$$= \frac{1}{2}\widehat{\mu}\frac{r(X_t^{*,x},t)}{X_t^{*,x}} - \frac{1}{2}\lambda^2\frac{1}{X_t^{*,x}}\left(\frac{1}{2}\frac{\partial}{\partial x}r^2(x,t)\Big|_{x=X_t^{*,x}}\right)$$

$$= \left(\frac{1}{2}\sigma^2 - \frac{1}{2}\widehat{\mu}r_x(X_t^{*,x},t)\right)\tilde{\pi}_t^{*,x}. \tag{97}$$

ii) The sensitivity of R_t^X with respect to σ is given by

$$\frac{\partial R_t^X}{\partial \sigma} := \left.\frac{\partial B\left(R,t;x\right)}{\partial \sigma}\right|_{R=R_t}$$

$$= \left(\frac{\lambda^2 T}{\sigma} r_x(x,0) - \frac{\lambda^2 (T-t)}{\sigma} r_x(X_t^{*,x},t) - \frac{2\lambda}{\sigma^2} R_t \right) \frac{r(X_t^{*,x},t)}{X_t^{*,x}}$$
$$= \left(\lambda T r_x(x,0) - \lambda (T-t) r_x(X_t^{*,x},t) - \frac{2}{\sigma} R_t \right) \tilde{\pi}_t^{*,x}. \quad (98)$$

iii) The sensitivity of R_t^X with respect to $\widehat{\mu}$ is given by

$$\frac{\partial R_t^X}{\partial \widehat{\mu}} := \left. \frac{\partial B\,(R,t;x)}{\partial \widehat{\mu}} \right|_{R=R_t}$$
$$= \left(\frac{t}{2} - \frac{\lambda T}{\sigma} r_x(x,0) + \frac{\lambda (T-t)}{\sigma} r_x(X_t^{*,x},t) + \frac{1}{\sigma^2} R_t \right) \frac{r(X_t^{*},t)}{X_t^{*,x}}$$
$$= \left((T-t) r_x(X_t^{*,x},t) - T r_x(x,0) + \frac{\sigma}{2\lambda} t + \frac{1}{\widehat{\mu}} R_t \right) \tilde{\pi}_t^{*,x}. \quad (99)$$

Example 7. Let the investor have CRRA utility (25) as in Example 1. Recall that $\pi_t^* = \frac{\lambda}{\sigma\gamma} X_t^*$, so that (94) yields that $\tilde{\pi}_t^{*,x} = \frac{\lambda}{\sigma\gamma}$. By (17) and (26), it is easily seen that $r(x,t) = \frac{1}{\gamma} x$. Then (95) and (96) yield

$$\frac{\partial R_t^X}{\partial R_t} = \frac{\lambda}{\sigma\gamma}, \qquad \text{and} \qquad \frac{\partial^2 R_t^X}{\partial R_t^2} = 0.$$

Furthermore, (97), (98) and (99), respectively, yield that

$$\frac{\partial R_t^X}{\partial t} = \frac{1}{2} \left(\frac{\widehat{\mu}}{\gamma} - \frac{\lambda^2}{\gamma^2} \right),$$
$$\frac{\partial R_t^X}{\partial \sigma} = \left(\frac{\lambda^2}{\gamma^2 \sigma} t - \frac{2\lambda}{\gamma \sigma^2} R_t \right)$$

and

$$\frac{\partial R_t^X}{\partial \widehat{\mu}} = \left(\left(\frac{1}{2\gamma} - \frac{\lambda}{\gamma^2 \sigma} \right) t + \frac{1}{\gamma \sigma^2} R_t \right).$$

Note that, for CRRA utility the cumulative excess return on the optimal wealth is linear with respect to the cumulative excess return on the stock. Moreover, the other sensitivities have the same monotonicity properties as their counterparts in Example 6.

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