Optimal Systemic Risk Mitigation in Financial Networks

Agostino Capponi* Peng-Chu Chen†

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Abstract

We propose a novel multi-period bilateral clearing framework, where the level of systemic risk is mitigated through an optimal bailout allocation strategy. The interbank liability network evolves stochastically over time, with default events having a persistent impact on the balance sheet structure of the network. The optimal bailout policy and associated clearing payments are recovered as the solution of a constrained stochastic dynamic programming problem. We develop a numerical analysis showing that optimal bailout allocations are able to reduce significantly the systemic risk level when liability exposures are heterogeneous and volatile. Our analysis provides a tool to support regulator decisions of when and where to intervene so to prevent the onset of potential threats to financial stability.

*School of Industrial Engineering, Purdue University, West Lafayette, IN, 47906. Email:capponi@purdue.edu

†School of Industrial Engineering, Purdue University, West Lafayette, IN, 47906. Email:chen621@purdue.edu

1 Introduction

Financial institutions are connected to each other via a sophisticated network of multilateral exposures. Through these linkages, distress or failure of a financial institution triggering large unexpected losses on its trades, can seriously affect the financial status of its counterparties, possibly leading them into default. The recursive interdependence in this network of exposures is typically referred to as *systemic risk*, and has been responsible for many failures experienced by mono-line insurers and investment banks during the crisis.

The intricate structure of linkages can be naturally captured via a network representation of the financial system. Such a network models the interlinking exposures between financial institutions, and can thus assist in detecting important shock transmission mechanisms. This is especially relevant in the current post-crisis regime, given a series of decisions taken by governmental authorities to better monitor systemically important entities. Moreover, as also mentioned in IMF (2010), most policy recommendations are targeted towards structural changes which can mitigate the adverse consequences emerging in such closely intertwined systems during times of crisis. Those include refinements in the lender of last resort principles, new funding liquidity and leverage restrictions for banks, as well as capital surcharges based on an institution's likely contribution to systemic risk, see also Staum (2012) for a survey.

Starting with the seminal paper by Allen and Gale (2001), who employed an equilibrium approach to model the propagation of financial distress in a credit network, many other approaches have recently been proposed to explain systemic risk. Gai and Kapadia (2010) use statistical techniques from network theory to model how contagion spreads via direct and indirect counterparty exposures, and analyze how the knock-on effects of distress at some financial institutions can force other entities to write down the value of their assets. Along similar lines, Cifuentes et al. (2005) consider fire sales of illiquid assets, and provide an equilibrium framework to compute prices for illiquid assets, as well as the firms' capital after accounting for losses due to illiquidity. Battiston et al. (2012a) describe the network time evolution using stochastic processes, and introduce the financial accelerator to characterize the feedback effect arising from changes in the financial conditions of an agent. Battiston et al. (2012b) demonstrate that systemic risk does not necessarily decrease if the connectivity of the underlying financial network increases.

Our paper belongs to the stream of literature generated from the seminal contribution of Eisenberg and Noe (2001), who developed a clearing system framework consistent with bankruptcy laws, to analyze systemic risk in interbanking networks. Such a framework was utilized and extended along several directions. Staum and Liu (2012) analyze how systemic risk in financial networks should be quantified and allocated to individual institutions. Rogers and Veraart (2012) relax the assumption made in Eisenberg and Noe (2001) that a defaulting bank can liquidate its assets at face value to repay due liabilities, and identify circumstances under which banks have incentives to rescue others. In the context of insurance, Blanchet and Shi (2012) consider a financial network involving two types of participants, insurance and reinsurance companies, and provide a model to capture the total losses generated from default cascades originated from a reinsurance company. The literature discussed so far has considered structural models of systemic risk. On the empirical side, Cont et al. (2012) and Angelini et al. (1996) have analyzed, respectively, Brazilian and Italian interbank systems, showing how defaults transmit through the payment system and originate systemic crisis.

The above mentioned studies have analyzed the consequences caused by defaults using a static model of counterparty exposures. Although static models provide insights about immediate consequences caused by defaults, they do not capture the propagation and aftershocks of default events. We advance the above literature by developing a *multi-period clearing* system, where the level of the systemic risk is controlled in each period so to optimize a target systemic loss over a fixed time horizon. Building on the original framework proposed by Eisenberg and Noe (2001), we consider an interbank liability structure which evolves stochastically, and explicitly model the persistent impact that the default of a node has on the network. The regulator, hereon referred to as *lender of last resort* (LLR), mitigates systemic risk by providing bailout loans to illiquid, but solvent nodes. Such an allocation policy is consistent with practice, as also discussed in Rochet and Vives (2004). We work under the hypothesis that the LLR has complete information on the liability structure of the network. We remark that even if such data is not readily available, it is possible to construct accurate proxies for interbank liabilities using standard methodologies, see for instance Docherty and Wang (2010). Such procedures use transaction data from the U.S. Federal Fund Markets as proxies for decomposing the observed total liabilities into interbank liabilities.

We reformulate the problem of finding an optimal clearing sequence within a Markov decision framework, and show that the optimal bailout policy and clearing payment sequence may be recovered as the solution of a constrained stochastic dynamic programming problem. We devise an an approximate stochastic dynamic programming methodology to efficiently solve the dynamic programming problem when the network is high dimensional. Our methodology is based on the rollout algorithm, see also Bertsekas and Tsitsiklis (1996), and selects a bailout plan using a suitable combination of a finite set of heuristics, each of them computing a suboptimal allocation policy.

We define two measures to capture the residual systemic risk and residual default rate in the network after controlling for the optimal rescue plan. By means of a numerical analysis, we show that high levels of systemic risk and default rates appearing in networks whose interbank liabilities are highly volatile and heterogeneous can be reduced significantly when an optimal bailout plan is provided.

The rest of paper is organized as follows. Section 2 introduces the building blocks of our framework. Section 3 develops the multi-period clearing payment system. Section 4 presents the construction of the clearing payment sequence. Section 5 performs a numerical analysis to assess systemic risk mitigation arising in specific network configurations. Section 6 concludes the paper.

2 The Framework

We define the structure of the financial network, and introduce basic notation and definition used throughout the paper.

2.1 Financial Network

We model the financial system as a digraph $G = (V, E)$, where the set V of nodes represents the financial firms, and the set E of edges the liability relations between nodes (a direct edge between node i and j indicates that i is a debtor of j). We fix a finite time horizon, divided into discrete intervals $[t, t+1)$, $t \in \{0, 1, \ldots, T-1\}$.

The state of the financial network is characterized by 3-tuple $(\mathbf{L}^t, \iota^t, \mathbf{a}^t)$, Here, $\mathbf{L}^t$ is the interbank *liability matrix* at t, with L^t_{ij} denoting the amount of liabilities owed by i to j at t. We use ι^t to denote the *operating cash inflow vector*, i.e. ι^t_i quantifies the proceeds generated from operation activities of node i at t. The vector $\mathbf{a}^\mathbf{t}$ is the *illiquid assets vector*, with a^t_i being the amount of illiquid assets held by node i, which he cannot readily convert into cash. Each node is initially endowed with a certain amount of illiquid assets, used to determine its solvency state, and only transferred as part of the default liquidation procedure to other nodes, as detailed later.

The interbank liabilities and operating cash inflows are modeled as discrete time stochastic processes, to capture the uncertain nature of the financial environment. We use $\mathbf{l}^t$ to denote the total liability vector, with $l_i^t = \sum_{j \neq i, j \in V} L_{ij}^t$ denoting the amount of total obligations of node i to all other nodes at t. Further, we denote by

$$\Pi_{ij}^t = \begin{cases} \frac{L_{ij}^t}{l_i^t} & \text{if } l_i^t > 0 \\ 0 & \text{if } l_i^t = 0, \end{cases}$$

the relative size of liabilities owed by i to j at t. Here, $\mathbf{\Pi}^t$ is the associated *liabilities proportion matrix*.

2.2 Lender of Last Resort (LLR)

We introduce an outside entity, whose goal is to provide bailout loans to illiquid yet solvent nodes, so to optimize the *expected flow of total payments* across the network.

A node is said to be *rescued* if it receives a bailout loan from the LLR. A node uses cash remaining after paying liabilities to his creditors in the network, to repay some (or all) of the bailout loan amount currently owed to the LLR.

We denote by $o_i^t \geq 0$ the bailout loan granted by the LLR to node i at t; $\mathbf{o}^t$ is the associated vector. Further, b_i^t denotes the amount of *repaid loan* by i at t. This leads to the following inductive relation for the amount q_i^t of *unrepaid loan* of node i by time t:

$$\begin{cases} q_i^0 = 0 \\ q_i^{t+1} = (q_i^t + o_i^t - b_i^t)(1 + r_c) & , t \in \{0, 1, \dots, T-1\}. \end{cases}$$

We distinguish between the interbank interest rate r from the rate r_c at which bailout loans need to be repaid. [1]

We denote by B the initial budget at disposal of the LLR. The latter will allocate it to financially distressed nodes according to a self-financing strategy. In other words, we require the following condition to hold

$$\sum_{i=1}^{n}(o_i^t + q_i^t) \leq B(1+r_c)^t \ , \ t \in \{0, 1, \ldots, T-1\}.$$

2.3 Illiquidity and Insolvency

We denote by p_i^t the total payment of node i at time t. We use $v_i^t \geq 0$ to denote the *available cash* to i at t. The following inductive relation allows computing v_i^{t+1} from v_i^t.

$$\begin{cases} v_i^0 = 0 \\ v_i^{t+1} = (1+r)\left(\sum_{j \neq i, i \in V} \Pi_{ji}^t p_j^t + \iota_i^t + v_i^t - l_i^t - q_i^t\right)^+ \quad , \ t \in \{0, 1, \ldots, T-2\}. \end{cases}$$

In words, the cash available to node i at time t, plus inflows coming from payments made by his debtors and cash generated from his operations, can be reinvested at the market rate, only after netting with the amount used to repay his creditors and the LLR. For brevity, we define

$$c_i^t = \sum_{j \neq i, i \in V} \Pi_{ji}^t p_j^t + \iota_i^t + v_i^t. \tag{2.1}$$

We impose that in each period, a previously rescued node which is currently liquid, solvent but still indebted to the LLR must decrease the owed amount to the maximum

[1] Indeed, all loans granted from the Fed under the emergency program were repaid with an interest rate ranging between 0.5% to 3.5%, which was different from the prevailing market rate.

possible extent. Concretely, we require

$$b_i^t = \min\{(c_i^t - l_i^t)^+, q_i^t\} \ , \ t \in \{0, 1, \dots, T-1\}. \tag{2.2}$$

We define a node $i \in V$ to be *illiquid* at time t if $c_i^t < l_i^t$. We use e_i^t to denote the *net assets* of node i at time t. This is given by

$$e_i^t = \sum_{\tau=t+1}^{T-1} (1+r)^{-(\tau-t)} \left(\sum_{j \neq i, j \in V} \mathbb{E}_t \left[L_{ji}^\tau \right] - \mathbb{E}_t \left[l_i^\tau \right] \right) - l_i^t + c_i^t + a_i^t.$$

Here, for a given random variable X, we denote by $\mathbb{E}_t[X]$ the conditional expectation of X given the information set available at time t.

A node $i \in V$ is said to be *solvent* at time t if $e_i^t \geq 0$.

2.4 Default

We describe the mechanism triggered upon default of a node. If node i defaults at t, its illiquid assets are immediately distributed to his creditors according to a proportionality rule. The market value of node i's illiquid assets is then detracted from the amount of total liabilities owed by i at the default time. The node i is then managed by a trustee from period $t+1$ to $T-1$. The trustee collects (1) payments that node i is supposed to receive from its debtors in the network, and (2) illiquid assets, possibly transferred to node i after the liquidation procedure completed by any other defaulted node. Both liquid and illiquid assets collected by the trustee are distributed to node i's creditors after the time horizon.

Definition 2.1. A node is said to *default* at time t if it is (1) illiquid and insolvent at t or (2) illiquid and solvent at t but not rescued by the LLR.

The *default indicator vector* at time t, denoted by $\mathbf{d}^t$, is defined as $d_i^t = 1$ if node

i defaulted at time $s < t$, and 0 otherwise. $\mathbf{d}^0$ is the zero vector. For future purposes, we denote by $\Delta \mathbf{d}^t := \mathbf{d}^{t+1} - \mathbf{d}^t$ the vector indicating the nodes defaulting in the time period $[t, t+1)$.

Next, we specify how the illiquid assets of a defaulted node are distributed to his creditors within the network. Suppose node i defaults at t. Then his illiquid assets are transferred based on the relative fraction of debt owed to his creditors, netted of the payment done at the default time. More specifically, define

$$\begin{aligned} w_{ij}^t &= \sum_{\tau=t}^{T-1} (1+r)^{-(\tau-t)} \mathbb{E}_t [L_{ij}^\tau] - \Pi_{ij}^t p_i^t \text{ for } i, j \in V \\ w_i^t &= \sum_{j \neq i, j \in V,} w_{ij}^t. \end{aligned}$$

Then, the illiquid assets of each node are governed by the following recursive equation

$$\begin{aligned} a_i^0 &:= a_0 \\ a_i^t &= (1+r) \left((1 - \Delta d_i^{t-1}) a_i^{t-1} + \Delta d_i^{t-1} (a_i^{t-1} - w_i^{t-1})^+ + \sum_{j \in V, j \neq i} \Delta d_j^{t-1} \frac{w_{ji}^{t-1}}{w_j^{t-1}} \min\{a_j^{t-1}, w_j^{t-1}\} \right) \end{aligned}$$

If i were not to default at $t-1$, its illiquid assets at t include the ones owned in the previous period, plus all assets distributed from his creditors who defaulted at $t-1$; otherwise, it includes the illiquid assets netted of the debt owed to his creditors in the previous period, plus the illiquid assets transferred from his creditors who defaulted at $t-1$.

3 The Multi-Period Clearing Payment System

We develop a multi-period clearing payment system, which generalizes the single period clearing system in Eisenberg and Noe (2001). In addition, we model the consequences that a default has on the future evolution of the network, and we allow the LLR to optimally control the size of unrepaid liabilities through the network. We ensure that in each time clearing payments satisfy the standard conditions imposed by bankruptcy laws: limited liability of equity, priority of liability over equity, and proportional repayments of liabilities in default. This leads to the following

Definition 3.1. Given a dynamic network, $\{(\mathbf{L}^t, \iota^t)\}_{t=0}^{T-1}$, a time sequence of 2-tuples $\{(\mathbf{p}^t, \mathbf{o}^t)\}_{t=0}^{T-1}$ is a *clearing sequence* if it satisfies the following conditions:

a. *Systemically efficient rescuing.* The LLR provides bailout loans to illiquid yet solvent nodes so to

$$\text{maximize}_{\{\mathbf{o}^t\}_{t=0}^{T-1}} \sum_{t=0}^{T-1} (1+r)^{-t} \sum_{i \in V} p_i^t.$$

b. *Proportional repayment of liabilities.* A node $i \in V$ pays $\Pi_{ij}^t p_i^t$ to node j at time t, were node i not to default before t.

c. *Absolute priority.* In each time t, if a node i does not default at time t, it pays in full its liabilities or, if it defaults at time t, it uses all of his available cash to pay his current creditors. If it defaults before time t, it does not make any payment. Formally, for $i \in V$, $t \in \{0, 1, \ldots, T-1\}$, we have

$$p_i^t = \begin{cases} l_i^t, & \text{if } d_i^t = 0 \text{ and } \Delta d_i^t = 0, \\ c_i^t, & \text{if } d_i^t = 0 \text{ and } \Delta d_i^t = 1, \\ 0, & \text{if } d_i^t = 1. \end{cases}$$

d. *Admissible bailout loans.* The LLR provides bailout loans only to illiquid yet solvent nodes, i.e. for $i \in V$ at time t,

$$\begin{cases} o_i^t > 0 & \Rightarrow c_i^t < l_i^t \text{ and } e_i^t \geq 0 \text{ and } d_i^t = 0, \\ o_i^t = 0 & \Leftarrow (c_i^t < l_i^t \text{ and } e_i^t < 0) \text{ or } d_i^t = 1. \end{cases}$$

e. *Effective bailout loans.* Nodes are rescued with the minimum needed amount, i.e.

$$o_i^t > 0 \Rightarrow o_i^t = l_i^t - c_i^t.$$

We remark that the admissible bailout loan condition above is consistent with Rochet and Vives (2004), where it is claimed that bailout rules of this type are typically followed by regulators.

4 Construction of Clearing Sequence

The systemically efficient rescuing requirement in the definition of a clearing sequence requires the development of a strategy which optimally decides when and how to allocate bailout loans. To this purpose, we introduce a *Markov decision* framework, and utilize stochastic dynamic programming techniques to recover clearing sequences.

4.1 Markov Decision Process

We consider a Markov decision process in a discrete-time finite horizon. Time is divided into discrete intervals called decision epochs that are indexed by $t \in \{0, 1 \ldots, T-1\}$. The t-th epoch corresponds to the interval $[t, t+1)$. We define $X^t = (\mathbf{L}^t, \iota^t) \in \mathcal{X}$ to be an exogenous stochastic process, whose first component is the realization of the matrix of liabilities in epoch $t-1$, and whose second component is the realization of the operating cash inflow vector in epoch $t-1$. The n-dimensional bailout loan vector, $\mathbf{o}^t \in \mathcal{O}^t$, represents the decision process with $\mathcal{O}^t$ being the feasible set of bailout loans. The set $\mathcal{X}$ is assumed countable and $\mathcal{O}^t$ is finite for $t \in \{0, 1 \ldots, T-1\}$. We then define the state at time t to be $s^t = (\mathbf{v}^t, \mathbf{a}^t, \mathbf{q}^t, \mathbf{d}^t) \in \mathcal{S}$, where $\mathcal{S}$ is countable. We recall that the first three components are n-dimensional vectors denoting, respectively, cash, illiquid assets, and unrepaid bailout loans associated to each node, while the last component is a n-dimensional vector indicating nodes defaulted by time t. In the beginning of decision epoch t, event occurrence follows the timeline illustrated in Figure 1. First, the state s^t is measured. Then, the stochastic process X^t is observed, and finally the decision $\mathbf{o}^t$ is taken.

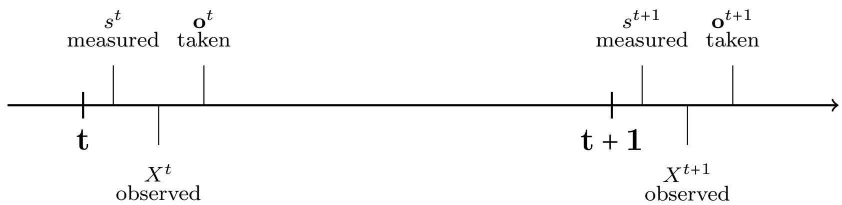

Figure 1: The time line indicates the sequence of event occurrence in epoch t.

Assume we are in epoch t, with triplet $(s^t, \mathbf{o}^t, X^t)$. Let $f(s^t, \mathbf{o}^t, X^t)$ be the vector-valued function yielding the state s^{t+1} at the beginning of decision epoch $t+1$. We define the lattice operation

$$\mathbf{x} \wedge \mathbf{y} := (\min\{x_1, y_1\}, \min\{x_2, y_2\}, \ldots, \min\{x_n, y_n\})$$

for two vectors, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Next, we illustrate how f maps from s^t to s^{t+1}.

$$f_1(s^t, \mathbf{o}^t, X^t) = s_1^{t+1} := \mathbf{v}^{t+1} = (1+r)(\mathbf{c}^t - \mathbf{l}^t - \mathbf{q}^t)^+,$$

$$f_2(s^t, \mathbf{o}^t, X^t) = s_2^{t+1} := \mathbf{a}^{t+1} = (1+r)\left((\mathbf{1} - \Delta\mathbf{d}^t) \cdot \mathbf{a}^t + \Delta\mathbf{d}^t \cdot (\mathbf{a}^t - \mathbf{w}^t)^+ + \left[\frac{w_{ij}^t}{w_i^t}\right]^\top (\Delta\mathbf{d}^t \cdot (\mathbf{a}^t \wedge \mathbf{w}^t))\right),$$

$$f_3(s^t, \mathbf{o}^t, X^t) = s_3^{t+1} := \mathbf{q}^{t+1} = (1+r_c)(\mathbf{q}^t + \mathbf{o}^t - \mathbf{b}^t),$$

$$f_4(s^t, \mathbf{o}^t, X^t) = s_4^{t+1} := \mathbf{d}^{t+1} = \mathbf{d}^t + \Delta\mathbf{d}^t,$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the component-wise product of vectors $\mathbf{x}$ and $\mathbf{y}$, $[\frac{x_{ij}}{x_i}]$ denotes the matrix whose (i,j)th entry is $\frac{x_{ij}}{x_i}$, and $(\mathbf{y})^+$ denotes the vector whose ith entry equals $\max\{0, y_i\}$, and $\mathbf{w}^t$ is a vector whose entries are given by w_i^t. Recall from Eq. (2.1) that $\mathbf{c}^t$ depends on $\mathbf{v}^t$ and X^t. Similarly, from Eq. (2.2), we have that $\mathbf{b}^t$ depends on $\mathbf{v}^t$, X^t, and $\mathbf{q}^t$. Hence, the state transition function is well defined. The state transition probability is given by

$$\mathbb{P}^s[\, s^{t+1} \big| \, s^t, \mathbf{o}^t] = \mathbb{P}^X \left[\omega : s^{t+1} = f\left(s^t, \mathbf{o}^t, X^t(\omega)\right)\right], \tag{4.1}$$

and represents the probability that the state at time $t+1$ is s^{t+1} conditional on the current state s^t, and the decision $\mathbf{o}^t$ taken at time t.

The feasible set of decision policies depends on s^t, and must verify the constraints indicated in the above definition of clearing sequence. Concretely, we define the constraint region

$$\begin{aligned}
\mathcal{O}^t(s^t) = \{\mathbf{o}^t : \quad & p_i^t = (1 - d_i^t)\min\{l_i^t, c_i^t + o_i^t\}, \\
& o_i^t > 0 \Rightarrow c_i^t < l_i^t \text{ and } e_i^t \geq 0 \text{ and } d_i^t = 0, \\
& o_i^t = 0 \Leftarrow (c_i^t < l_i^t \text{ and } e_i^t < 0) \text{ or } d_i^t = 1, \\
& o_i^t > 0 \Rightarrow o_i^t = l_i^t - c_i^t \\
& \sum_{i \in V}(q_i^t + o_i^t) \leq B(1 + r_c)^t \text{ for } i \in V\},
\end{aligned} \tag{4.2}$$

where the first constraint indicates that the bailout loan should satisfy the absolute priority requirement in definition 3.1, the second to fourth constraints indicate that only illiquid yet solvent nodes can be rescued, and with the minimum needed amount. The last constraint is the budget constraint of the LLR.

Let $z^t(s^t, \mathbf{o}^t, X^t)$ be the single period total payment function at time t when the state is s^t, the decision taken is $\mathbf{o}^t$, and the realized process is X^t. This is given by

$$z^t(s^t, \mathbf{o}^t, X^t) = \sum_{i \in V} p_i^t := \sum_{i \in V} p_i^t(s^t, \mathbf{o}^t, X^t).$$

Our objective is to maximize the expected total discounted sum of payments, over the time horizon across all feasible policies $\pi = (\mathbf{o}_\pi^0, \mathbf{o}_\pi^1, \dots, \mathbf{o}_\pi^{T-1})$. Denoting the optimal value by $Z^0(s^0)$ and the set of all feasible policies by $\Pi = \mathcal{O}^0 \times \mathcal{O}^1 \times \cdots \times \mathcal{O}^{T-1}$, we have the total payment function at time 0 is given by

$$Z^0(s^0) = \max_{\pi \in \Pi} \mathbb{E}\left[\sum_{\tau=0}^{T-1} (1+r)^{-\tau} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau)\right]. \tag{4.3}$$

We next prove the existence of an optimal policy to Eq. (4.3), which can be found using the principle of dynamic programming.

Proposition 4.1. *There exists at least one feasible policy to Eq.* (4.3)*.*

Proof. We claim that $\mathbf{o}_\pi^t = \mathbf{0}$ for $t \in \{0, 1, \dots, T-1\}$ is a feasible policy. Given any state s^t, it is obvious that $\mathbf{0}$ satisfies the second to fourth constraints in Eq. (4.2). For all $i \in V$, by definition, $q_i^0 = b_i^0 = 0$. By induction, $q_i^t = b_i^t = 0$ for all t, hence $\mathbf{0}$ satisfies the last constraint in Eq. (4.2), as $\sum_{i \in V}(q_i^t + o_i^t) = 0 \leq B(1+r_c)^t$ for all t. It remains to show that $\mathbf{0}$ also satisfies the first constraint in Eq. (4.2). We next prove that there exists a vector $\mathbf{p}^t$ satisfying the first constraint. This is equivalent to showing that the mapping

$$\Phi(\mathbf{p}^t) := \left[\operatorname{diag}(\mathbf{1} - \mathbf{d}^t)(\mathbf{\Pi}^{t\top}\mathbf{p}^t + \iota^t + \mathbf{v}^t)\right] \wedge \mathbf{l}^t$$

has a fixed point. Because Φ is the composition of two positive increasing functions, respectively $\mathbf{y} \to \operatorname{diag}(\mathbf{1} - \mathbf{d}^t)(\mathbf{\Pi}^{t\top}\mathbf{y} + \iota^t + \mathbf{v}^t)$ and $\mathbf{y} \to \mathbf{y} \wedge \mathbf{l}^t$, it must be a positive increasing function on $\mathbf{p}^t$. By Tarski's fixed point theorem, the set of fixed points of Φ is not empty; hence, there exists at least one vector $\mathbf{p}^t$ satisfying the first constraint in Eq. (4.2). This concludes the proof. $\square$

When a bailout allocation $\{\mathbf{o}^t\}$ is specified, the sequence $\{\mathbf{p}^t\}$ is uniquely determined provided that the subgraph of the financial network induced by the set of non-defaulted nodes satisfies the regularity condition introduced in Eisenberg and Noe (2001). A simple sufficient condition guaranteeing a sequence of regular subgraphs induced by non-defaulted nodes is that all nodes have positive operating cash inflows at all time t.

Lemma 4.2. *Let the financial network be such that the subgraphs induced by the non-defaulted nodes are regular for all t. Then, if a bailout allocation $\{\mathbf{o}^t\}$ is specified, the sequence $\{\mathbf{p}^t\}$ is uniquely determined.*

Proof. Consider a generic epoch t. We need to show that there exists a unique solution $\mathbf{p}^t$ to the first system of equalities in (4.2). The first constraint in Eq. (4.2) may be rewritten as

$$p_i^t = \begin{cases} \min\{l_i^t, \sum_{j\neq i, j\in V} \Pi_{ji}^t p_j^t + \iota_i^t + v_i^t + o_i^t\}, & \text{if } d_i^t = 0, \\ 0, & \text{if } d_i^t = 1. \end{cases}$$

Let $i \in \{k \in V | d_k^t = 0\}$. Since $\mathbf{o}^t$ is specified, ι_i^t and v_i^t are known, then we can define $\iota_i^t + v_i^t + o_i^t$ as $\hat{\iota}_i^t$ and simplify the first equation as

$$p_i^t = \min\left\{l_i^t, \sum_{\substack{j\neq i \\ j\in\{k\in V|d_k^t=0\}}} \Pi_{ji}^t p_j^t + \hat{\iota}_i^t\right\}.$$

Since the subgraph induced by set $\{k \in V | d_k^t = 0\}$ is regular, it was shown in Eisenberg and Noe (2001) that the above system of equations has a unique solution, p_i^t. □

Let

$$Z^t(s^t) = \max_{\substack{\mathbf{o}_\pi^\tau \in \mathcal{O}^\tau \\ \forall t\leq\tau\leq T-1}} \mathbb{E}\left[\sum_{\tau=t}^{T-1} (1+r)^{-(\tau-t)} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau) \middle| s^t, \mathbf{o}_\pi^t\right].$$

Following, we show that $Z^t(s^t)$ satisfies the Bellman equation.

Lemma 4.3. *$Z^t(s^t)$ satisfies the Bellman equation.*

Proof.

$$
\begin{aligned}
&Z^t(s^t) \\
&= \max_{\substack{\mathbf{o}_\pi^\tau \in \mathcal{O}^\tau \\ \forall t \le \tau \le T-1}} \mathbb{E}\left[\sum_{\tau=t}^{T-1} (1+r)^{-(\tau-t)} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau) \middle| s^t, \mathbf{o}_\pi^t \right] \\
&= \max_{\substack{\mathbf{o}_\pi^\tau \in \mathcal{O}^\tau \\ \forall t \le \tau \le T-1}} \left\{ z^t(s^t, \mathbf{o}_\pi^t, X^t) + (1+r)^{-1} \mathbb{E}\left[\sum_{\tau=t+1}^{T-1} (1+r)^{-(\tau-t-1)} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau) \middle| s^t, \mathbf{o}_\pi^t \right] \right\} \\
&= \max_{\substack{\mathbf{o}_\pi^\tau \in \mathcal{O}^\tau \\ \forall t \le \tau \le T-1}} \Big\{ z^t(s^t, \mathbf{o}_\pi^t, X^t) + (1+r)^{-1} \\
&\qquad\qquad \mathbb{E}\left[\mathbb{E}\left[\sum_{\tau=t+1}^{T-1} (1+r)^{-(\tau-t-1)} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau) \middle| s^{t+1}, \mathbf{o}_\pi^{t+1} \right] \middle| s^t, \mathbf{o}_\pi^t \right] \Big\} \\
&= \max_{\mathbf{o}_\pi^t \in \mathcal{O}^t} \Big\{ z^t(s^t, \mathbf{o}_\pi^t, X^t) + (1+r)^{-1} \\
&\qquad\qquad \mathbb{E}\left[\max_{\substack{\mathbf{o}_\pi^\tau \in \mathcal{O}^\tau \\ \forall t+1 \le \tau \le T-1}} \mathbb{E}\left[\sum_{\tau=t+1}^{T-1} (1+r)^{-(\tau-t-1)} z^\tau(s^\tau, \mathbf{o}_\pi^\tau, X^\tau) \middle| s^{t+1}, \mathbf{o}_\pi^{t+1} \right] \middle| s^t, \mathbf{o}_\pi^t \right] \Big\} \\
&= \max_{\mathbf{o}_\pi^t \in \mathcal{O}^t} \left\{ z^t(s^t, \mathbf{o}_\pi^t, X^t) + (1+r)^{-1} \mathbb{E}\left[Z^{t+1}(s^{t+1}) \middle| s^t, \mathbf{o}_\pi^t \right] \right\}. \qquad (4.4)
\end{aligned}
$$

The third equation comes from the law of iterated expectations. The fourth equation follows from the Markovian property of the decision process. This concludes the proof. □

For future purposes, we refer to the quantity $z^t(s^t, \mathbf{o}_\pi^t, X^t) + (1+r)^{-1}\mathbb{E}\left[Z^{t+1}(s^{t+1}) \middle| s^t, \mathbf{o}_\pi^t \right]$ in Eq. (4.4) as the *Q-factor* at time t, and to the second term $(1+r)^{-1}\mathbb{E}\left[Z^{t+1}(s^{t+1}) \middle| s^t, \mathbf{o}_\pi^t \right]$ as the *cost-to-go* function at time t.

5 Computing the Optimal Bailout Strategy

A financial network usually consists of a large number of nodes. This indicates that the above defined stochastic dynamic programming problem may become computational intractable as the state space grows exponentially, making it unfeasible to compute the Q-factor. In this section, we develop a suboptimal solution approach, and use techniques from neuro-dynamic programming to provide an efficient solution to the problem, see Bertsekas and Tsitsiklis (1996). Given the network state in each time, such techniques employ heuristic policies and Monte-Carlo simulation to construct the Q-factor approximation. Then, they apply the so-called rollout algorithm to select the suboptimal policy.

5.0.1 Q-factor Approximation

In each decision epoch t, given the state s^t and the realized stochastic process X^t, the k-th heuristic computes the Q-factor approximation for the optimal policy through the following steps.

Step 1. Compute the value $z_k^t(s^t, \mathbf{o}_{\pi_k}^t, X^t)$ by heuristic k and the next state by $s_{\pi_k}^{t+1} = f(s^t, \mathbf{o}_{\pi_k}^t, X^t)$, where $\mathbf{o}_{\pi_k}^t$ is the bailout loan selected by policy k.

Step 2. Generate M random paths $(X_m^{t+1}, X_m^{t+2}, \ldots, X_m^T), m = 1, \ldots, M$.

Step 3. Compute the value $z_{k,m}^\tau(s_{\pi_k,m}^\tau, \mathbf{o}_{\pi_k,m}^\tau, X_m^\tau)$ and the simulated state trajectories $s_{\pi_k,m}^{\tau+1} = f(s_{\pi_k,m}^\tau, \mathbf{o}_{\pi_k,m}^\tau, X_m^\tau)$ from $\tau = t+1$ to $T-1$, $m = 1, \ldots, M$.

Step 4. Denote by $H_k^t(s^t, X^t)$ the Q-factor approximation derived from heuristic k given by

$$H_k^t(s^t, X^t) = z_k^t(s^t, \mathbf{o}_{\pi_k}^t, X^t) + (1+r)^{-1}\mathbb{E}\left[Z_k^{t+1}(s_{\pi_k}^{t+1}) \middle| s^t, \mathbf{o}_{\pi_k}^t \right]$$

$$=z_k^t(s^t, \mathbf{o}_{\pi_k}^t, X^t) + \sum_{\tau=t+1}^{T-1} (1+r)^{-(\tau-t)} \frac{1}{M} \sum_{m=1}^{M} z_{k,m}^{\tau}$$

5.0.2 Heuristics

We consider one myopic and two non-myopic heuristics to produce the Q-factor approximations. Myopic heuristics only consider the present state of the network, while the non-myopic heuristics account for expected values of future states of the network.

(1) *Myopic Heuristic:* It ignores the cost-to-go function and only computes the solution which maximizes the single period payment function. For each decision epoch t, given s^t and X^t let us denote the solution and objective value derived by this method as $\mathbf{o}_{\pi_1}^t$ and z_1^t respectively. They are given by

$$\mathbf{o}_{\pi_1}^t = \arg\max_{\mathbf{o}^t \in \mathcal{O}^t} z^t(s^t, \mathbf{o}^t, X^t) \qquad z_1^t = \sum_{i \in V} p_i^t(s^t, \mathbf{o}_{\pi_1}^t, X^t).$$

We call $H_1^t(s^t, X^t)$ the Q-factor approximation obtained using this heuristic.

Our choice of this heuristic is motivated by financial and optimality considerations. From a financial perspective, greedy approaches are in line with behavior of regulators. As discussed in Hoggarth et al. (2004), under systemic conditions the objective of the latter is to restore financial stability immediately. The myopic method captures this notion, as it maximizes the liquidity injected into the network as early as possible. From an optimality perspective, the myopic method may perform better than other greedy-type algorithms if liabilities and operating cash inflows in the network have a homogeneous composition.

Definition 5.1. $\{(\mathbf{L}^t, \iota^t)\}_{t=0}^{T-1}$ is *homogeneous* if for each t, L_{ij}^t has the same probability law as $L_{k,l}^t$, for any pairs $(i,j) \neq (k,l)$, and ι_i^t has the same law as ι_j^t, for $i \neq j$.

Given a homogeneous network, the expected values of future liabilities and operating cash inflows are the same for all nodes; this means that rescuing one node is not necessary better than rescuing any other. However, if more liquidity is injected earlier into the network, it can generate a higher flow of future payments as it stays longer into the system. Hence, maximizing the liquidity pumped into the network in each time may be more effective than using other greedy criteria.

(2) *Too-Big-To-Fail Heuristic:* It rescues illiquid yet solvent nodes according to balance sheet size (nodes with largest balance sheet values are rescued first), until the bailout budget is exhausted. For decision epoch t, given s^t and X^t, we perform the following

Step 1. Assign $\mathbf{o}^t = \mathbf{0}$, and compute

$$p_i^t = \begin{cases} l_i^t, & \text{if } d_i^t = 0 \text{ and } \Delta d_i^t = 0, \\ c_i^t, & \text{if } d_i^t = 0 \text{ and } \Delta d_i^t = 1, \\ 0, & \text{if } d_i^t = 1, \end{cases} \quad \text{for } i \in V.$$

This step computes the payments that will be made by each node if the bailout loan is not provided.

Step 2. Compute c_i^t and net asset e_i^t for each node i. Recall from Eq. (2.1) that c_i^t depends on $\mathbf{p}^t$, $\mathbf{v}^t$ and X^t. Identify illiquid yet solvent nodes as satisfying $c_i^t < l_i^t$ and $e_i^t \geq 0$.

Step 3. Compute the balance sheet size of node i as

$$\left[\sum_{\tau=t+1}^{T}\left((1+r)^{-(\tau-t)}\sum_{j\neq i, j\in V}\mathbb{E}_t\left[L_{ji}^{\tau}\right]\right)+c_i^t+\gamma a_i^t\right]+\left[\sum_{\tau=t}^{T}(1+r)^{-(\tau-t)}\mathbb{E}_t\left[l_i^{\tau}\right]\right],$$

where the first term represents the amount of total assets, while the second term the amount of total liabilities.

Step 4. Denote the bailout loan vector by $\mathbf{o}_{\pi_2}^t$. Rank-order the nodes in decreasing order of balance sheet value. Following the rank, provide the bailout loan $o_{\pi_2,i}^t = l_i^t - c_i^t$ to each node i, until all the nodes have been rescued or the bailout loan budget is exhausted.

Step 5. The objective value z_2^t is given by $z_2^t = \sum_{i\in V} p_i^t(s^t, \mathbf{o}_{\pi_2}^t, X^t)$.

We call $H_2^t(s^t, X^t)$ the Q-factor approximation obtained using this heuristic.

(3) *Reward-based Heuristic:* The LLR rescues the illiquid yet solvent nodes in decreasing order of outstanding liabilities until the bailout budget is exhausted. The size of outstanding liabilities is taken as a measure of the potential amount of payments the node can generate in the future, if rescued. For the decision epoch t, given s^t and X^t, the liability size of each illiquid yet solvent node i is computed as

$$\sum_{\tau=t}^{T}(1+r)^{-(\tau-t)}\mathbb{E}_t\left[l_i^{\tau}\right].$$

We denote the the bailout strategy by $\mathbf{o}_{\pi_3}^t$ and the objective value by z_3^t . We call $H_3^t(s^t, X^t)$ the Q-factor approximation obtained using this heuristic.

This method exploits the fact that for each node, the total payment over the time horizon is a non-decreasing function of the total liabilities over the same horizon. Such a method performs well if the financial network is *heterogeneous*, where heterogeneous means not homogeneous. Consider for example a configuration consisting of (1) illiquid nodes whose present value of outstanding liabilities is large, and are expected to perform significantly better in terms of future operating cash inflow, (2) nodes with smaller outstanding liabilities but predicted to continue performing poorly in future cash inflows. Under this setup, rescuing the nodes with larger outstanding liabilities can generate larger payment flows in the network, given that payments are increasing in the amount of outstanding liabilities and operating cash inflows. Consequently, this method is expected to perform better than the myopic method.

5.0.3 Rollout Algorithm

We apply the *rollout* algorithm to improve the performance of the original heuristics. This yields the suboptimal bailout policy as well as the corresponding clearing payment sequence. We remark that a similar approach has been followed by Bertsekas et al. (1997), in the context of a machine maintenance and repair problem. While the rollout algorithm in Bertsekas et al. (1997) is based on the approximate cost-to-go function, here it is based on the approximate Q-factor. Denote the suboptimal policy and corresponding bailout strategy in decision epoch t by $\tilde{\pi}^t$ and $\tilde{\mathbf{o}}^t$, respectively. The rollout algorithm may be described as follows. For $t = 0, \dots, T-1$

Step 1. Compute the Q-factor approximation, $H_k^t(s^t, X^t)$, for $k = 1, 2, 3$.

Step 2. Select the suboptimal policy $\tilde{\pi}^t$ associated to the heuristic with the largest H_k^t, along with the corresponding bailout loans, i.e.

$$\tilde{\pi}^t = \arg\max_{k \in \{1,2,3\}} H_k^t(s^t, X^t) \qquad \tilde{\mathbf{o}}^t = \mathbf{o}^t_{\tilde{\pi}^t}.$$

Not restricted to the three heuristics mentioned in the previous section, in general, the rollout algorithm improves the performance of the original heuristics, i.e.

Lemma 5.2. *Consider K heuristics used by the rollout algorithm,. Then*

$$\mathbb{E}\left[\sum_{\tau=0}^{T-1}(1+r)^{-\tau}z^{\tau}(s^{\tau},\mathbf{o}_{\bar{\pi}^{\tau}}^{\tau},X^{\tau})\right] \geq \mathbb{E}\left[\sum_{\tau=0}^{T-1}(1+r)^{-\tau}z^{\tau}(s^{\tau},\mathbf{o}_{\pi_k}^{\tau},X^{\tau})\right] \text{ for } k \in \{1,\ldots,K\}.$$

Proof. The proof directly follows from Bertsekas (2005), see their Proposition 3.1. □

6 Systemic Risk Analysis

We provide an analysis to assess the systemic risk reduction obtained after controlling for the optimal bailout policy. To this purpose, we first introduce two measures of systemic risk in the context of our framework. We then use the approximate dynamic programming method illustrated in the previous section to analyze these measures under different network configurations.

6.1 Systemic Risk Measures

The first measure is the *residual systemic risk*, defined as

$$RS = \mathbb{E}\left[\frac{\sum_{t=0}^{T-1}(1+r)^{-t}\left(\sum_{i\in V} l_i^t - z^t(s^t,\mathbf{o}^t,X^t)\right)}{\sum_{t=0}^{T-1}(1+r)^{-t}\sum_{i\in V} l_i^t}\right].$$

which gives the percentage of unrepaid liabilities within the network, after accounting for optimal bailout policy provided by the lender of last resort. Recall that $\mathbf{d}^T$ is the default indicator vector by the end of the time horizon. The second measure is the *residual default rate*, defined as

$$RD = \frac{1}{n}\mathbb{E}\left[\sum_{i\in V} d_i^T\right],$$

i.e. the percentage of defaulted nodes, after controlling for the optimal bailout loan policy. Clearly, if the regulator policy is effective, these measures should be small.

6.2 Experimental Setup

We fix the number of network nodes to $n = 20$, and the time horizon to $T = 20$. The time step used in the simulation is $\Delta t = 1$, which results in twenty payment periods. Liabilities between each pair of nodes are assumed to follow a multi-period binomial model. The amount of liabilities $L_{ij}^{t+\Delta t}$, owed by i to j at time $t + \Delta t$, is given by uL_{ij}^t with probability $\mathbb{P}_u$ and by δL_{ij}^t with probability $\mathbb{P}_\delta$. Here $u \geq 1$, $\delta \leq 1$, and $\mathbb{P}_\delta = 1 - \mathbb{P}_u$. We also assume that the operating cash inflow of each node follows a multi-period binomial model. In order to analyze the quality of the optimal selection policy returned by our algorithm, we consider a scenario where illiquid assets are large enough that each node is always solvent, hence qualified to be rescued if illiquid. We consider two network configurations, homogeneous and heterogeneous. For each configuration, the initial amount of due liabilities from each node is \$500. In the homogeneous case, each node is liable to all others in the network. In the heterogeneous case, the network is divided into three blocks, one large block and two small blocks. The large block consists of twelve nodes, all liable to each other. Each small block consists of four nodes, each of which only liable to higher indexed nodes in the block. The initial amount of liabilities owed by each node to his creditors is \$500. The parameters u and δ associated to the homogeneous and heterogeneous configuration are given in Table 1. For both configurations, the operating cash inflows generated by each node have initial sizes of \$200, and then evolve independently following a binomial model with $u = 1.1$ and $\delta = 0.9$.

Figure 2 illustrates the initial snapshot for the two network configurations.

Following the definition of Bernoulli distribution, we set $\mathbb{P}_u = 0.5 - \sqrt{0.25 - \sigma^2}$.

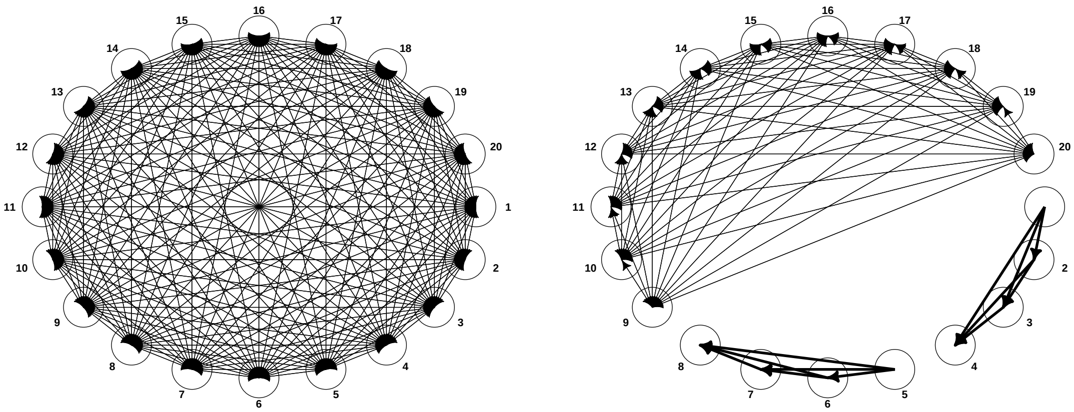

Figure 2: The left panel shows the homogeneous network at time $t = 0$. The right panel shows the heterogeneous network at time $t = 0$.

Market interest rate, r	8%
Bailout loan interest rate, r_c	0.75%
Liquidation discounted factor, γ	80%

Binomial Movements	Homogeneous	Heterogeneous Large	Heterogeneous Small
u	1.1	1.1	$1 + 0.1/t$
δ	0.9	0.9	$(1 - 0.125t)^+$

Table 1: The left table shows the economic parameters. The right table shows the liability movement ratios used for the nodes belonging to different subgraphs following the binomial model at time t

Clearly, $\mathbb{P}_u$ increases when σ increases, thus results in higher expected value for the total liabilities in the network. This also reflects the fact that higher volatility of liabilities is commonly associated with higher trading volumes within the network, thus leading to larger liabilities within the network.

We fix the number of Monte-Carlo runs to 50. We compare the performance of the base heuristics described in the previous section, and of the the rollout algorithm. For the latter, in each time time period, each base heuristic is evaluated on one-hundred Monte-Carlo paths, to approximate the Q-factor. To facilitate a comparison with the scenario where no mitigation is applied, we also superimpose the No-Bailout curves to report systemic risk and default rate of the networks under the case when no bailout loan is provided.

6.3 Economic Analysis

Our analysis demonstrates that *homogeneous are more robust than heterogeneous networks* to systemic events. As it appears from Figure 3, the homogeneous network requires smaller amount of bailout loans to mitigate systemic risk. For example, to obtain a 20% level of systemic risk, it is necessary to provide an amount of $\$30,000$ in bailout loans for the homogeneous configuration, versus an amount of $\$600,000$ needed in the heterogeneous case. Notice that this happens, despite the fact that the total amount of liabilities generated in the homogeneous network amount to 7.8 million, almost twice as large as the 3.7 million amount in the heterogeneous network. The conclusion is that the homogeneous configuration induces smaller illiquid amounts over time, and hence it is more robust against default events with respect to the heterogeneous network.

Figures 3 - 5 also show that *d*efault rate behaves similarly to systemic risk. This implies that the number of defaulted nodes and total payments are also highly correlated in the multi-period model. It further suggests that for homogeneous networks, greedy-type algorithms rescuing as many nodes as possible in each iteration, can produce good suboptimal allocation policies to our problem. This happens because the state of nodes is similar across time. Consequently, the sooner the bailout loan is provided, the smaller is the amount of unrepaid liabilities generated over time, thus lowering systemic risk.

Next, we analyze how bailout budget, volatility of liabilities, and correlation across liabilities, affects the level of systemic risk.

6.3.1 Impact of Bailout budget

We fix $\sigma = 0.5$. We vary the budget from \$10,000 to \$50,000 in the homogeneous configuration, and from \$400,000 to \$600,000 in the heterogeneous one, given that the total size of network liabilities is higher in the homogeneous configuration. As expected, Figure 3 is consistent with intuition that systemic risk decreases when higher bailout loans are available. We remark that it is possible to achieve the same systemic risk reduction in both configurations, although a much smaller budget is used by the LLR in the homogeneous case. When the network is heterogeneous, the rollout algorithm outperforms all other heuristics by a significant amount. When the configuration is homogeneous the network state over time does not change dramatically, hence myopic strategies also tend to offer near optimal bailout policy allocations.

6.3.2 Impact of Correlation

We assume the liability exposures are pairwise correlated with identical coefficient ρ as describe next. Denote by $\mathbb{P}_{uu}$ the joint probability that liability exposures of i to j and of k to l simultaneously increase, i.e.

$$\mathbb{P}_{uu} = \mathbb{P}(L_{ij}^{t+\Delta t} = uL_{ij}^{t}, L_{kl}^{t+\Delta t} = uL_{kl}^{t}) \qquad (i,j) \neq (k,l).$$

Similarly, we denote by $\mathbb{P}_{\delta\delta}$, $\mathbb{P}_{u\delta}$, and $\mathbb{P}_{\delta u}$ the joint probability that two node's liability exposures move down and in opposite directions. From the definition of Bernoulli distribution, we have

$$\mathbb{P}_{uu} = \mathbb{P}(L_{ij}^{t+\Delta t} = uL_{ij}^{t})\mathbb{P}(L_{kl}^{t+\Delta t} = uL_{kl}^{t} | L_{ij}^{t+\Delta t} = uL_{ij}^{t}) = \rho\mathbb{P}_u(1 - \mathbb{P}_u) + \mathbb{P}_u{}^2 \tag{6.1}$$

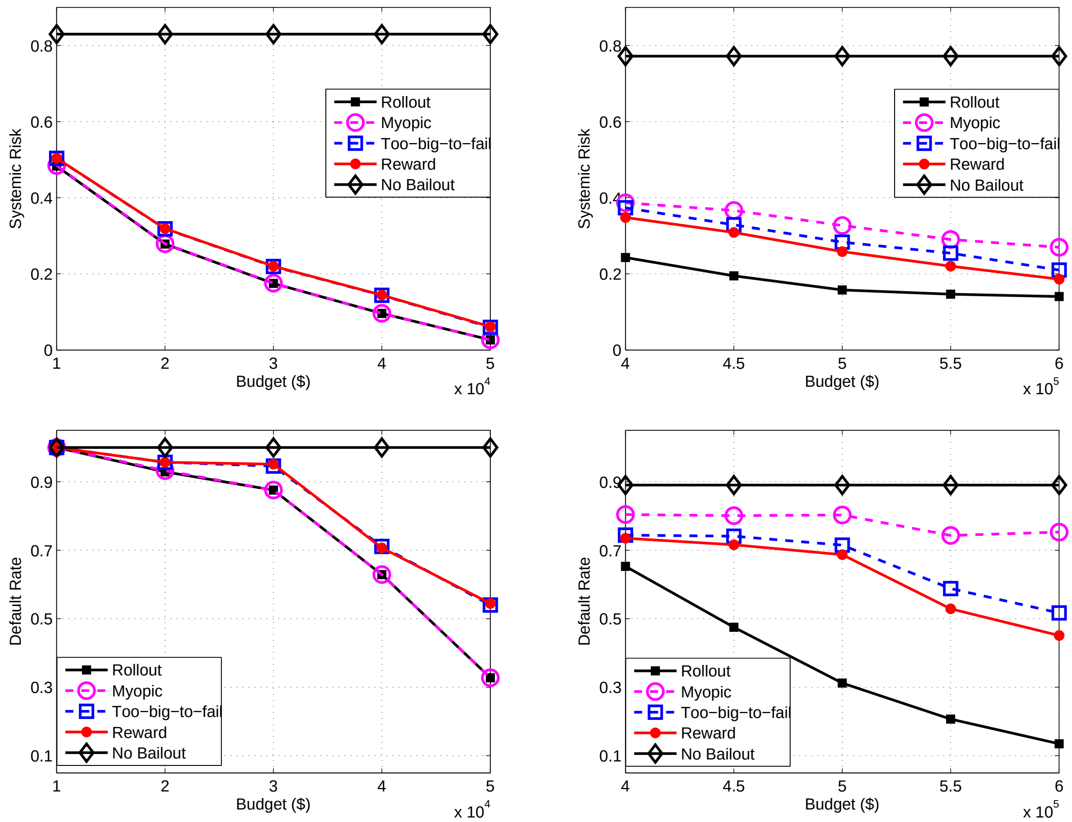

Figure 3: The left panels show the residual systemic risk and residual default rate each algorithm derives for the homogeneous network across different budgets. The right panels show the results for the heterogeneous network.

Similarly, we obtain

$$\mathbb{P}_{u\delta} = \mathbb{P}_{\delta u} = (1-\rho)(1-\mathbb{P}_u)\mathbb{P}_u \qquad \mathbb{P}_{\delta\delta} = 1 - \mathbb{P}_{uu} - 2\mathbb{P}_{u\delta}$$

Clearly, as ρ increases, $\mathbb{P}_{uu}$ and $\mathbb{P}_{\delta\delta}$ increase while $\mathbb{P}_{u\delta}$ decreases. This results in closer co-movement of the node's liability exposures within the network.

We fix the volatility $\sigma = 0.5$. We set $B = \$30,000$ in the homogeneous case, and $B = \$500,000$ in the heterogeneous case. We vary ρ from 0 to 0.9.

Figure 4 indicates that, in the absence of mitigation, systemic risk increases with correlation among liability exposures (the No-bailout curves decrease under both configurations). This happens because increases in correlation make the liability structure more homogeneous given that liability exposures are more likely to move in the same direction. Consequently, this results in smaller systemic risk. The effect is more significant in the homogeneous configuration, where a 65% reduction is achieved, against a 45% reduction achieved in the heterogeneous case.

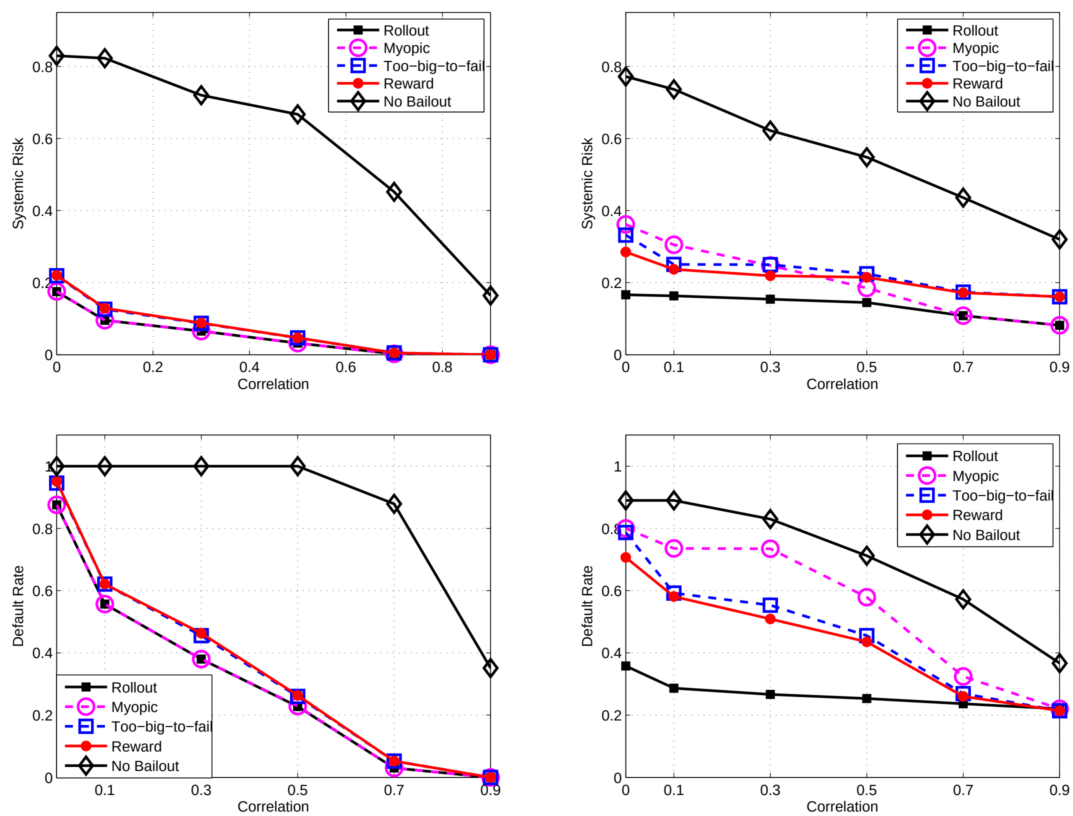

Figure 4: The left panels show the residual systemic risk and residual default rate each algorithm derives for the homogeneous network across different correlations. The right panels show the results for the heterogeneous network.

6.3.3 Impact of Volatility

We fix B = \$5,000 in the homogeneous case, and B = \$100,000 in the heterogeneous case. We then vary the volatility from 0.1 to 0.4. Since $\mathbb{P}_u$ is increasing in σ, the network would result in higher total liabilities as σ increases. To control for that, and only analyze the impact of larger variations between incoming and outgoing liabilities across nodes, we choose the initial liability amounts so that the expect total amount of interbank liabilities across time is the same, regardless of the volatility level. Figure 5 indicates that higher volatility in liability exposures increase the systemic risk level. This happens because higher volatility generates larger differences between incoming and outgoing liabilities, When no bailout loan is provided, the effect is the strongest, with larger number of nodes of defaulting, and consequently resulting in the highest systemic risk levels. Bailout loan mitigation alleviates, but not fully remove this effect. Indeed, the residual systemic risk keeps increasing with volatility. Such an increase, not only generates higher illiquid amounts for each illiquid node, but also increases the number of illiquid nodes. The effect is more pronounced in the homogeneous configuration. As the volatility becomes sufficiently high (for instance 0.2 in our benchmark case), the network loses most of its homogeneity and systemic risk increases rises faster. We remark that this mechanism exhibits similarity with counterparty valuation adjustments, which have been demonstrated to be very sensitive to volatility exposures, see Capponi (2012) for an illustration, and identified as the major drivers of the the systemic risk crisis.

6.4 Computational Analysis

We evaluate the performance of each algorithm, and observe the following facts.

(1) *Myopic algorithm performs better in homogeneous settings.* For heterogeneous

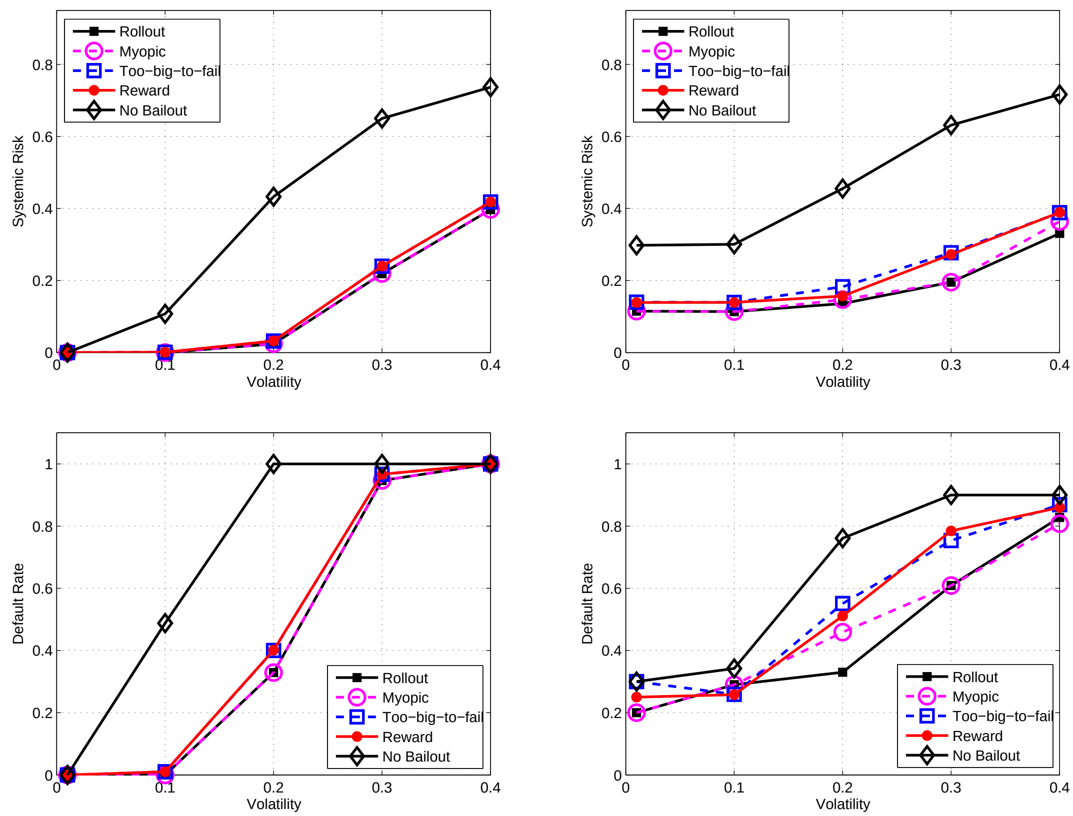

Figure 5: The left panels show the residual systemic risk and residual default rate each algorithm derives for the homogeneous network across different volatilities. The right panels show the results for the heterogeneous network.

networks, the myopic algorithm performs worse than too-big-to-fail and reward-based algorithm. The last two always rescue the nodes with larger balance sheet or liability sizes, which have the potential to generate higher payments with respect to nodes rescued by the myopic algorithm. In homogeneous networks, the too-big-to-fail and reward-based algorithms no longer have the advantage we just described due to the similarity of balance sheet structure across nodes. Consequently, the myopic algorithm outperforms the other two, given that it maximizes the total payments in each time. This also shows that the *systemically* *important nodes are not necessarily the ones with the largest balance sheet size.*

(2) *Rollout algorithm performs better under both network configurations.* Figure 3 – 5 and Table 2 show that the rollout algorithm has always a better systemic risk mitigation effect with respect to the other three heuristics. It always improves upon the performance of the other three methods, with the improvement being more significant in the heterogeneous network. Indeed, for a heterogeneous network with $B = \$500K, \sigma = 0.5, \rho = 0$, the rollout algorithm is able to lower the systemic risk to about half of the systemic risk level achieved by the other methods.

Budget	10,000	20,000	30,000	40,000	50,000
Rollout	0.4832	0.2781	0.1749	0.0959	0.0264
Myopic	0.4845	0.2791	0.1755	0.0963	0.0266
Reward	0.5037	0.3188	0.2202	0.1442	0.0613
Too-big-to-fail	0.5035	0.3184	0.2192	0.1437	0.0595

Volatility	0.01	0.1	0.2	0.3	0.4
Rollout	0	0	0.0240	0.2189	0.3967
Myopic	0	0	0.0320	0.2397	0.4180
Reward	0	0.0006	0.0322	0.2400	0.4182
Too-big-to-fail	0	0	0.0245	0.2195	0.3973

Correlation	0	0.1	0.3	0.5	0.7	0.9
Rollout	0.1749	0.0955	0.0655	0.0323	0.0022	0
Myopic	0.1755	0.0956	0.0655	0.0323	0.0022	0
Too-big-to-fail	0.2192	0.1264	0.0872	0.0469	0.0051	0
Reward	0.2202	0.1291	0.0879	0.0471	0.0052	0

Table 2: Residual systemic risk in the homogeneous network

7 Conclusions

We developed a multi-period clearing payment system building on the framework originally proposed by Eisenberg and Noe (2001). We have modeled the systemic consequences of a default event both on current and future evolution of the financial network. Our framework allows the systemic risk to be optimally controlled by a default-free regulator, under a clearing payment system consistent with standard laws of bankruptcy. After reformulation as a Markov decision problem, we have shown that optimal bailout policy and corresponding clearing payment sequence can be recovered as the solution of a constrained stochastic dynamic programming problem.

We analyzed the power of our framework in predicting residual systemic risk and default rate under two interbank network configurations, namely homogeneous and heterogeneous. We find that the networks with higher correlated and lower volatile liabilities, or more homogeneous liability structure, are more robust to default events. Comparisons between the suboptimal bailout policy recommended by our algorithm and alternative bailout policies suggested by regulators shows that too-big-to-fail or greedy policies may result in significantly higher levels of systemic risk.

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