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Systemic Risk: The Dynamics under Central Clearing

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Abstract

We develop a tractable model to resemble asset value processes of financial institutions trading with the central clearinghouse for risk mitigating purposes. Each institution allocates assets between its loan book and the account used to trade with the central clearinghouse. We show that a unique equilibrium allocation profile arises when institutions adjust trading positions to perfectly hedge risk stemming from their loan books. We then analyze the dynamic equilibrium path. As a regulatory monitoring tool, our model shows a buildup of systemic risk, manifested through the increase of market concentration, whose negative size externalities can be internalized via a self-funding systemic risk charge mechanism. We provide new testable predictions, including that (i) the volatility of the traded portfolio of a member can be forecasted by the collective capital committed by all others, (ii) hedging becomes increasingly costly for an institution as its asset value increases, (iii) market shocks have smaller impact on allocation decisions than operational shocks.

Introduction

Banks create size externalities. Large banks are more likely to be complexly structured and deeply interconnected with other banks and the financial system. Resolution of large failed financial institutions is harder for regulators to deal with because they generate higher negative externalities compared to small institutions. Such too-big-to-fail

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systemic risk concerns motivated government sponsored bailouts of banks during the Great Recession and are currently primary regulatory concerns.

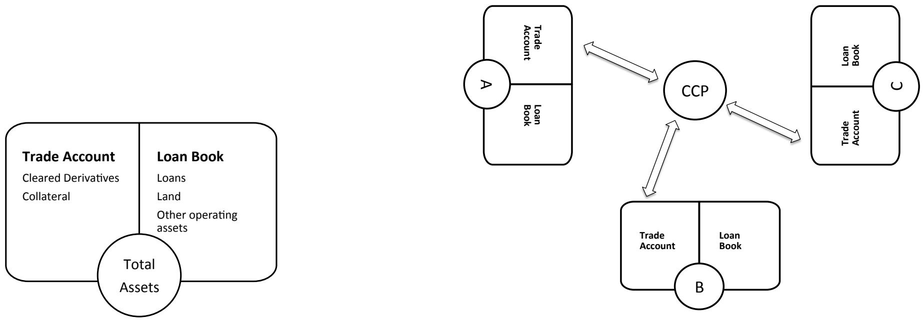
The most significant regulatory response to the Great Recession was the Dodd–Frank Wall Street Reform and Consumer Protection Act (2010), which mandated the central clearing of all standardized over-the-counter (OTC) derivatives. Central clearing of OTC derivatives has grown also due to dealers' interest in facing robust counterparties. As a result, dealers' trading motives are driven less by speculation and more by a need to hedge risks incurred in the course of business.[1] In the centralized clearing framework, clearinghouses act as central counterparties (CCPs) in both the exchange-traded and OTC markets. Large banks participate as clearing members of the clearinghouse.

We investigate the formation of too-big-to-fail systemic risk in such a regulated market using the Herfindahl–Hirschman Index, a measure of market concentration. We construct a model for the asset value processes of clearing members that trade with the central clearinghouse for risk-mitigating hedging purposes only. In particular, banks trade to hedge undesired risk — that risk that members are inherently exposed to, but unprepared to manage. Our model shows that systemic risk can build up over time. We show that the volatility of the traded portfolio of a member depends on both its committed trading capital and the capital of all remaining clearing members. Importantly, a unique equilibrium allocation of capital between trading and nontrading activities arises, determined by all members' assets and hedging preferences. Properties of this equilibrium allocation imply that increases in market concentration can accumulate over time.

We provide several testable predictions: (i) Hedging is increasingly costly (in terms of committed capital) as the asset value of a member increases. Large banks have to commit proportionally larger amounts of trading capital than small banks; (ii) volatility of the traded portfolio can be forecasted not only by the committed capital of the member, but also by the collective committed capital of all other members; (iii) shocks to the financial market have a much smaller effect on asset allocation decisions than operational shocks (shocks to their business operations); (iv) capital raising and centralized trading have opposite effects on market concentration; (v) increases in market concentration are observed when financial institutions choose diverse business operations.

Our study uses a similar categorization of collateral demand as in Duffie et al. (2015). While they use historical asset returns and bilateral exposures to estimate collateral demand, we take collateral demand as given and infer from it market beliefs about future asset returns. In our model, members allocate assets between trading (the trade account) and nontrading (the loan book) activities (see Figure 1). They have hedging desires: their objective is to offset identified undesired risk, which has constant returns

[1] The Dodd–Frank Act restricts the trader functions of commercial banks to hedging purposes. This is commonly referred to as the "Volcker rule" (section 619 of the Dodd-Frank Act), and is essentially a ban on proprietary trading of banks. Exemptions from the ban include underwriting, market making-related activities, risk-mitigating hedging, trading in certain government obligations, certain trading activities of foreign banking entities, other permitted activities, and clarifying exclusions. Aside from risk-mitigating hedging, all other exemptions relate to a bank's function as a broker/dealer rather than a trader. See also Fed (2013).



(a) Clearing member allocates assets (b) Centralized trading

Figure 1: (1a) A clearing member always allocates its assets between its trade account and loan book. The trade account contains assets traded and used to trade with the CCP. The loan book contains loans and other assets needed for business operations. (1b) Clearing members A, B, and C use their trade account assets to trade with the CCP. They reallocate assets after profits and losses are realized.

to scale in loan book size, stemming from the loan book (see Figure 2).

We show that the riskiness of the trade account is implied both by the level of capital the member commits, reflecting the riskiness of positions chosen, and the level of capital the other members commit, reflecting the overall market risk. Thus, to perfectly hedge against undesired risk members must take into account the hedging needs of all market participants. We prove there exists a unique feasible allocation profile in which no member has an incentive to change its allocation. This yields the endogenous dynamics of members' asset values processes, i.e. accounting for their equilibrium asset allocation.[2]

To the best of our knowledge our trading model is the first symmetric information model that explicitly incorporates the zero-sum nature of financial trading. There is rich literature on trading originated by the seminal paper of Kyle (1985), in which price movements and trade volumes of securities are driven by asymmetric information and heterogeneous beliefs of traders. In contrast, in our model trades are motivated by the innate hedging desire of each member, and there is a single agreed-on price for each contract.

We first analyze the impact of trading on market concentration. As pointed out by the International Monetary Fund in its April 2014 Global Stability Report (IMF (2014)), concentration in the banking sector has increased even after the Great Recession for many countries, in part due to the governmental support of bank consolidations. In 2012, assets of the three largest banks in the United States represented 44 percent of total banking assets. We show this concentration phenomenon can be further exacerbated by the reformed, centralized trading setting. When loan books of members are exposed to

[2]Our model exhibits similarities with Uzawa's two sector model on economic growth (Uzawa (1961)). While he solves for allocations of labor and capital inputs and analyzes the deterministic equilibrium path, we solve for intra-temporal equilibrium allocations of I members and analyze the stochastic equilibrium path.

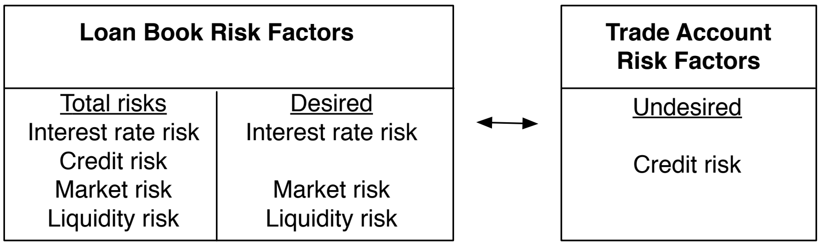


Figure 2: A member who views credit risk as undesired hedges its credit risk exposures with centrally cleared derivatives. The risk factors that it is innately exposed to come from the composition of its his loan book. Trade account positions are taken to offset undesired risk.

similar risk factors, loan book values co-move positively, resulting in little impact on market concentration. When they are exposed to diverse risk factors, however, market concentration can increase. This points to a trade-off between financial market diversity and systemic risk. While diverse loan book exposures can result in lower volatility in the aggregate asset value, they pose systemic risk concerns.

We then proceed to analyze how concentration accumulates over time. As mentioned above, concentration can arise from uncorrelated shocks; however, one would expect that these shocks can just as well reduce concentration. Interestingly, we find that the collective act of hedging undesired risk, while on the individual level is risk-mitigating and desirable, also leads to a buildup of systemic risk. Because large banks have large risks (in dollar amount) and their trade account volatilities have decreasing returns to scale in asset size, they invest a large proportion of assets into hedging (trading). Since the trade account assets are naturally hedged by the undesired risk factors stemming from the members' loan books, this high amount of asset value is hoarded by the large bank and preserves market concentration. Our study takes a benefit of the doubt approach to member behavior, in that systemic risk is not driven by asymmetric information, members' moral hazard (they carefully evaluate and hedge their risks), and inadequate risk management of the clearinghouse (collateral requirements are diligent). That systemic risk builds up is a feature of the *system*, rather than a feature of the individual agent behavior.

Exogenous inflows of capital can also affect market concentration. If small members choose to raise capital at a faster rate than large members, market concentration stemming from centralized trading can be reduced. However, growth in banks' assets is usually funded by debt as opposed to equity issuance (see Adrian et al. (2012)), hence resulting in increased leverage ratios. In addition, the government bailout of large banks has created an implicit big bank subsidy (see Santos (2014), Acharya et al. (2014)). This increases the costs of capital raising for small banks and hence indicates that it is unlikely for capital raising to reduce concentration.

The emergence of size externalities induced by the seemingly innocuous act of hedging has important policy implications. While hedging undesired risk may be optimal on an individual risk management level, preventive policies targeting concentration effects are needed on the systemic level. We indicate how our model can incorporate a systemic risk charge based policy which can be used to control market concentration. Concretely, we show that a Pigouvian tax charge proportional to members' trade account values, in conjunction with a trading mandate, reduces market concentration. A Pigouvian tax here is the the application of a charge on clearing member size, which in turn may relieve some systemic risks associated with central clearing.

Our model may be best viewed as a regulatory monitoring tool. We do not explicitly consider members' default, but rather focus on the time period before the occurrence a default event to study the buildup of systemic risk.[3] In this respect, our study is most closely related to monitoring approaches of Duffie (2014b) and Acharya et al. (2010a). Duffie (2014b) proposes a qualitative framework in which regulators use stress tests on systemically important financial firms to help evaluate the centrality of each firm in the financial network. Our model provides asset value dynamics which take into account the effects of centralized trading, and can thus supplement the quantitative aspects of these stress tests. Acharya et al. (2010a) propose a taxing system in which financial entities are taxed based on the extent and likelihood of their contribution to systemic risk. Our model provides a similar systemic risk charge. In both cases, the systemic externalities of large financial firms are internalized.

Our study contributes to the rapidly growing stream of literature on central clearing. Previous works have analyzed optimal determination of margins (Glasserman et al. (2014)) and specification of collateral requirements (Cumming and Noss (2013)). Differently from these studies which consider static models and focus on risk management and design of a clearinghouse, we propose a dynamical model to analyze systemic implications *after* a clearinghouse has been well designed. Duffie et al. (2015) study the impact on collateral demand of central clearing. We show that while in the cross section individual collateral demand is monotonic with respect to the total asset value of each member, marketwide collateral demand is not. Duffie and Zhu (2011) analyze the netting benefits resulting from central clearing and their impact in reducing counterparty risk. Biais et al. (2014) study the optimal design of derivative contracts and clearing mechanisms so that central clearing insures against counterparty risk. Duffie (2014a) discuss policies for defaults resolution, including the haircutting of variation margin gains, tear-ups, as well as the interruption of clearing services.

The rest of the paper is organized as follows. Section 1 develops the model. Section 2 studies the dynamics of our measure of market concentration, the Herfindahl–Hirschman Index. Section 3 presents a self-funding systemic risk charge policy targeting at reducing systemic risk. We discuss testable predictions of our model in section 4. Section 5 concludes. Appendix A provides supplementary notes on the dynamics of the trade account value. All technical proofs are delegated to appendix B.

[3]We note that the absence of default in our model does not contradict the need of collateral. It is the *potential* of default that calls for collateral, rather than actual defaults. An example of this phenomenon in a binomial economy is given in Geanakoplos (2010). In fact, equilibrium contracts traded in binomial economies require collateral just tight enough to eliminate default, as shown in Fostel and Geanakoplos (2015).

1 Model

The economy consists of I market participants, referred to as members, who novate all trades to a single clearinghouse. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbb{Q})$, where the filtration satisfies the usual conditions of completeness and right continuity (see Protter (2004)). Here, $\mathbb{Q}$ is a risk-neutral measure associated with the risk-free money-market account numéraire, so that value processes of all traded assets denominated in units of the numéraire are $\mathbb{Q}$-martingales. In what follows, we use $\circ$ to denote the component-wise (Hadamard) product. Let $t_m, m = 0, 1, 2, \ldots,$ be an equally spaced sequence of times, and $\Delta t = t_{m+1} - t_m$ be the length of each time period.

Member i has a *loan book* and a *trade account*, whose value processes are denoted by $L_i, M_i \geq 0$, respectively. The total asset value process of member i is denoted by $A_i := M_i + L_i$. The trade account contains centrally cleared derivatives positions along with the associated committed capital. The loan book contains all remaining assets of the member, including deposits, corporate loans, and other operating assets.

We assume the market is frictionless and treat all derivatives as contracts that carry zero value. This uniform treatment greatly simplifies our analysis and does not result in any loss of generality. Indeed, whenever a member purchases a derivative maturing at time T that has price P (in units of the numeraire security), the trade is equivalent to going long a futures contract at price P on the derivative maturing at time T, and committing P units of the numeraire security as collateral. All collateral requirements are met with the risk-free numeraire.[4]

The rest of the section is organized as follows. We analyze the dynamics of the trade account value in section 1.1. Section 1.2 develops the continuous time model of members' asset value dynamics.

1.1 Dynamics of trade account value

We discuss the components of collateral demand in section 1.1.1. This yields the trade account value dynamics studied in section 1.1.2. Throughout the paper, we will use bold symbols to denote vectors and matrices so to distinguish them from scalar quantities.

1.1.1 Components of collateral demand

Member i's collateral demand consists of two components: the initial margin requirement and the variation margin buffer. The initial margin is collateral posted to cover losses incurred by the clearinghouse when the member defaults and its outstanding positions need to be closed out. The variation margin buffer is the precautionary stock of collateral members set aside to meet potential variation margin payments.[5] Each member commits

[4]The International Swaps and Derivatives Association reports that, as of December 31, 2013, significantly increasing proportions of margin requirements are met by cash. Cash represented 61 percent and 99 percent of total collateral delivered to meet initial and variation margin requirements, respectively. See ISDA (2014), table 12.

[5]This categorization follows the approach outlined in Duffie et al. (2015). We do not differentiate between the precautionary variation margin buffer and the variation margin velocity drag. In a fully

exactly enough capital to satisfy its collateral demand. The member keeps sufficient initial margin, C_i, in its margin account and sets aside variation margin buffer V_i.

The clearinghouse practices *portfolio margining* and sets the initial margin requirement depending on the outstanding portfolio traded with the member. If member i has a derivatives portfolio $\mathcal{P}_i$ with the clearinghouse, we denote by $C_i = C_i(\mathcal{P}_i)$ the amount of collateral that the clearinghouse requires it to keep in a margin account. Consistent with anecdotal evidence, this collateral demand function is not member specific, i.e. two members with identical portfolios must have the same collateral demand.

We model the variation margin buffer as a constant fraction of the initial margin, $V_i := \lambda C_i$. Our choice is driven by the fact that the variation margin buffer is kept to cover daily changes in portfolio values, while the initial margin is designed to cover changes over a liquidation period, typically of five to ten business days. The total collateral demand, and thus committed capital, is $(1+\lambda)C_i$.

We assume that members maintain the buffer $V_i(t)$ at the minimum level that guarantees that the probability of losses exceeding the buffer is zero in a small time period of length Δt.[6] Thus the variation margin buffer is a measure of *maximal probable losses* that can result from holding the portfolio $\mathcal{P}_i$ (in a time period of length Δt). It is important to notice that V_i plays a dual role in our analysis, as it is both a measure of committed capital and of probable losses that can result from holding the portfolio $\mathcal{P}_i$.[7]

1.1.2 Trade account value dynamics

This section derives the trade account value dynamics under the risk-neutral probability measure in two steps. First, we use the collateral demand processes to infer the risk-neutral probabilities. We then give the dynamics under this measure.

We model the variations in market value of the positions of each member as follows. At time t, let $\{\mathcal{P}_i(t)\}_{i=1}^{I}$ be the portfolios held by members so that the market clears (the chosen portfolio positions sum up to zero). They commit capital equal to their collateral demands $\{C_i(t) + V_i(t)\}_{i=1}^{I}$. At time $t + \Delta t$ realization of market shocks results in gains for some members and losses for others. Member i receives payoff $R_i(t + \Delta t) \geq -V_i(t)$, since the loss that can occur in the period Δt is bounded above by $V_i(t)$. Since derivatives trading is a zero-sum game, $\sum_{i=1}^{I} R_i(t + \Delta t) = 0$ and $V(t) := \sum_{i=1}^{I} V_i(t) = \sum_{i=1}^{I} (V_i(t) + R_i(t + \Delta t))$. Thus, variations in market value can be represented by redistribution of aggregate buffer value $V(t)$ across members.

centralized trading setting, the drag component in their collateral model can be accounted for by multiplying the precautionary buffer by a constant. Differentiation thus has no qualitative effect on the analysis.

[6]According to Duffie et al. (2015), the variation margin buffer is typically set so that the probability of losses exceeding V_i is low within each day.

[7]To simplify the presentation, we assume that the initial margin requirement of the clearinghouse is the same as the maintenance margin requirement, so that members can withdraw collateral from their margin accounts when making gains, and make variation margin payments when suffering losses. Specifying the difference does not change the qualitative conclusions of our analysis but introduces complications when book-keeping the components of collateral demand.

We model this redistribution process by sectioning $V(t)$ into $N \in \mathbb{N}$ ticks of equal size, $\frac{V(t)}{N}$, and describe their ownership with categorical random variables. Define the categorical random variable $\chi_n(t+\Delta t) = i$ if the n-th tick is given to member i at time $t+\Delta t$. The *value at hand*, defined as the previous collateral demand plus the realized payoff, for member i at time $t+\Delta t$ is thus

$$C_i(t) + V_i(t) + R_i(t+\Delta t) = C_i(t) + \frac{V(t)}{N} N_i(t+\Delta t), \tag{1.1}$$

where $N_i(t+\Delta t) := \left|\{\chi_n(t+\Delta t) = i\}_{n=1}^{N}\right|$.

We now assume that these N categorical random variables are i.i.d. and define $q_i(t) := \mathbb{Q}(\chi_n(t+\Delta t) = i|\mathcal{F}_t)$. When N is large, this assumption is analogous to shocking all members' portfolio values with zero mean normal random variables, conditioned on them summing to zero, reflecting realistic stress testing scenarios employed by clearinghouses.[8]

Under the risk-neutral measure, the expected value at hand is equal to the cost of purchasing the portfolio $\mathcal{P}_i$, which is simply the amount of committed capital because we treat all derivatives as contracts with zero price. Thus,

$$\underbrace{(1+\lambda)C_i(t)}_{\text{committed capital}} = E^{\mathbb{Q}}\left[C_i(t) + \frac{V(t)}{N} N_i(t+\Delta t)\middle|\mathcal{F}_t\right].$$

We can now infer the risk-neutral probabilities $\{q_i\}_{i=1}^{I}$,

$$\frac{V_i(t)}{V(t)} = E^{\mathbb{Q}}\left[\frac{1}{N} N_i(t+\Delta t)\middle|\mathcal{F}_t\right] =: q_i(t).$$

[8]Notice that the independence assumption does not rule out observed correlation in identified risk factors, but merely puts a constraint on the extremity, or the "fat-tailed"-ness of portfolio movements. For example, consider the situation with two members, where at time t member 1's portfolio is long one contract, and member 2 is short the same contract. Assume there are four states of the world at time $t+\Delta t$, $\Omega = \{(0,0),(0,1),(1,0),(1,1)\}$, which occur with equal probability. The payoff of the portfolio to member 1, X_1, is

$$X_1(\omega) = \begin{cases} \$2, \omega = (1,1) \\ \$1, \omega \in \{(1,0),(0,1)\} \\ \$0 \ \ \omega = (0,0) \end{cases}.$$

We can model this by setting the tick size to \$1, and let

$$\chi_1(t+\Delta t) = \begin{cases} 1, \omega_1 = 1 \\ 2, \omega_1 = 0 \end{cases},$$

$$\chi_2(t+\Delta t) = \begin{cases} 1, \omega_2 = 1 \\ 2, \omega_2 = 0 \end{cases}.$$

An observer does not know the structural factors (ω_1, ω_2) which govern the payoffs, but identifies two risk factors, $F_1 := \frac{1}{3}\chi_1 + \frac{2}{3}\chi_2$ and $F_2 := \frac{2}{3}\chi_1 + \frac{1}{3}\chi_2$. These factors can explain the movements in the contract's value perfectly but are obviously correlated. Thus, while underlying structural shocks are independent, we can still observe correlation between identified risk factors.

We then obtain that $\mathbf{N}(t) = (N_1(t), \ldots, N_I(t))$ is distributed according to the following multinomial probability distribution function:

$$\mathbb{Q}(\mathbf{N}(t+\Delta t) = \mathbf{n}|\mathcal{F}_t) = N! \prod_{i=1}^{I} \frac{1}{n_i!} \left(\frac{V_i(t)}{V(t)}\right)^{n_i} \tag{1.2}$$

Next, we derive the multi-period dynamics of the trade account value in the limiting case, when the number of ticks goes to infinity and the length of the time period becomes infinitesimally small. For large N and small Δt, denote $\sigma'^{-2} := N\Delta t$. Then we can approximate the multinomial distribution given by Eq. (1.2) with a normal distribution.[9] Denote the conditional correlation matrix associated with the multivariate distribution in Eq. (1.2) by $\mathbf{\Sigma}(\mathbf{V}_i(t))$. The exact expression for $\mathbf{\Sigma}$ is given in appendix A. We can thus model the payoffs as

$$\begin{aligned}
&\mathbf{R}(t_{m+1}) \approx \sigma' \sqrt{\mathbf{V}(t_m) \circ (V(t_m)\mathbf{1} - \mathbf{V}(t_m))}\mathbf{Z}^{\mathbf{\Sigma}}_{m+1} \qquad (1.3)\\
&\mathbf{1} := (1, 1, \ldots, 1)' \in \mathbb{R}^I,\\
&\mathbf{Z}^{\mathbf{\Sigma}}_{m+1}|\mathcal{F}_{t_m} \sim \mathcal{N}(0, \mathbf{\Sigma}(\mathbf{V}(t_m))\Delta t)\\
&V(t_m) := \sum_{i=1}^{I} V_i(t_m)
\end{aligned}$$

Member i's trade account value, M_i, is given by $M_i := C_i + V_i = \frac{1+\lambda}{\lambda} V_i$. The gains stemming from the trade account $R_i(t_{m+1}) =: \Delta_m M_i$ are thus

$$\begin{aligned}
&\Delta_m \mathbf{M} = \sigma \sqrt{\mathbf{M}(t_m) \circ (M(t_m)\mathbf{1} - \mathbf{M}(t_m))} \circ \mathbf{Z}^{\mathbf{\Sigma}}_{m+1}, \qquad (1.4)\\
&\mathbf{Z}^{\mathbf{\Sigma}}_{m+1}|\mathcal{F}_{t_m} \sim \mathcal{N}(0, \mathbf{\Sigma}(\mathbf{M}(t_m))\Delta t).
\end{aligned}$$

Here, $\sigma = \sigma' \frac{\lambda}{1+\lambda}$.

1.2 Trading as a hedging mechanism

In this section we use the trade account value dynamics given by Eq. (1.4) to build a symmetric information model for asset value processes of members. Each member trades to hedge its undesired risk. We obtain a continuous time model when the size of the trading period shrinks to zero.

1.2.1 Total asset and loan book dynamics

The vector of changes in asset value of each member, denoted by $\Delta_m \mathbf{A} := \mathbf{A}(t_{m+1}) - \mathbf{A}(t_m)$, is given by:

$$\Delta_m \mathbf{A} = \boldsymbol{\phi} \circ \mathbf{A}\Delta t + \Delta_m \mathbf{M} + \Delta_m \mathbf{L}, \tag{1.5}$$

[9] When N is large, the multinomial distribution given by Eq. (1.2) converges weakly to a normal distribution.

In the above expression, $\boldsymbol{\phi}$ is a vector whose i-th entry is the *capital raising rate* of the i-th member, $\Delta_m\mathbf{M}$ is the vector of members' gains from trading, and $\Delta_m\mathbf{L}$ the vector of gains stemming from the loan book. At time t_m, each member allocates its entire asset value between its loan book and trade account. At time t_{m+1}, each member realizes gains or losses in its loan book and trade account, and additionally receives capital at rate $\boldsymbol{\phi}(t)$ from outside investors. Then, each member reallocates its current asset value between its loan book and trade account.

Our model for loan book gains is chosen to be

$$\begin{aligned} \Delta_m\mathbf{L} &= \theta\mathbf{L}(t_m) \circ \mathbf{Z}^{\boldsymbol{\Xi}}_{m+1}, \\ \mathbf{Z}^{\boldsymbol{\Xi}}_{m+1}|\mathcal{F}_{t_m} &\sim \mathcal{N}(0, \boldsymbol{\Xi}\Delta t). \end{aligned} \tag{1.6}$$

Here, $\mathbf{Z}^{\boldsymbol{\Xi}}_m$ is the vector of risk factors driving the loan books of members. Moreover, $\theta > 0$ is the loan book volatility assumed to be identical across members. Homogeneity is assumed since all clearing members are large financial institutions and hence the risk profiles of their operational activities tend to be similar.

Due to their own business models, members choose to expose themselves to specific levels and types of risk. We decompose the risk factors driving the loan book values, $\mathbf{Z}^{\boldsymbol{\Xi}}_m$, into desired and undesired risk factors as

$$\begin{aligned} \mathbf{Z}^{\boldsymbol{\Xi}}_{m+1} &= -\rho\mathbf{Z}^{\boldsymbol{\Sigma}}_{m+1} + \sqrt{1-\rho^2}\mathbf{Z}^{\boldsymbol{\Psi}}_{m+1}, \\ \mathbf{Z}^{\boldsymbol{\Psi}}_{m+1}|\mathcal{F}_{t_m} &\sim \mathcal{N}(0, \boldsymbol{\Psi}\Delta t). \end{aligned} \tag{1.7}$$

Here, $\rho \in (0,1)$ is the *hedging desire* parameter capturing the fraction of risk which is undesired for the member. Concretely, $-\mathbf{Z}^{\boldsymbol{\Sigma}}$ is the vector of members' *undesired risk.*[10] The i-th component of $\mathbf{Z}^{\boldsymbol{\Psi}}$ gives the desired portion of risk factors to which member i is exposed. We refer to $\mathbf{Z}^{\boldsymbol{\Psi}}$ as the vector of members' *desired risk.* Notice that this construction means that the members, in view of their undesired risk exposures, construct their trade accounts to perfectly hedge undesired risk, i.e. construct a trade account whose value process is driven by $\mathbf{Z}^{\boldsymbol{\Sigma}}$. We assume that the desired risk of each member is independent of its undesired risk, but not necessarily independent of the undesired risk of other members.

1.2.2 Hedging strategies and the equilibrium allocation

Variations in the loan book value originate from exposures to either undesired or desired risk factors. Indeed, using the decomposition in Eq. (1.7) along with Eq. (1.6), we may rewrite the changes in loan book values of members as

$$\Delta_m\mathbf{L} = \underbrace{-\rho\theta\mathbf{L}(t_m) \circ \mathbf{Z}^{\boldsymbol{\Sigma}}_{m+1}}_{\text{undesired price movements}} + \underbrace{\sqrt{1-\rho^2}\theta\mathbf{L}(t_m) \circ \mathbf{Z}^{\boldsymbol{\Psi}}_{m+1}}_{\text{desired price movements}}. \tag{1.8}$$

[10] To be precise, this is the undesired risk that is *traded.* Market clearing implies that for any traded undesired risk factor of a given member, there is a party willing to take on this risk. The undesired risks are thus correlated.

For notational convenience, we describe the allocation strategy of member i using the *trade ratio* process

$$\kappa_i := \frac{M_i}{A_i}. \tag{1.9}$$

Since both loan book and trade account values are nonnegative, $\kappa_i \in [0,1]$.

We next impose the fundamental behavioral assumption: *each member allocates assets to be perfectly hedged against all undesired price movements.* By a direct comparison of equations (1.4) and (1.8), this means that member i chooses a trade ratio $\kappa_i(t_m)$ so that

$$\begin{cases} -\rho\theta L_i(t_m) + \sigma\sqrt{M_i(t_m)(M(t_m) - M_i(t_m))} = 0 \\ M_i(t_m) = \kappa_i(t_m)A_i(t_m) \\ L_i(t_m) = (1-\kappa_i(t_m))A_i(t_m) \end{cases}. \tag{1.10}$$

We also impose a *market clearing* condition that the quantities of traded risk must satisfy:

$$\sum_{i=1}^{I} \sqrt{M_i(t_m)(M - M_i(t_m))}Z^{\mathbf{\Sigma}}_{i,m+1} = 0. \tag{1.11}$$

This condition states that the financial market merely transfers risk from one member to another, and the "excess demand" of exposure to any risk factor is zero. We show in appendix A that, given our choice of correlation matrix $\mathbf{\Sigma}$, this condition is indeed satisfied for any nonnegative process $\mathbf{M}$ (Proposition A.3). Thus changes in the mark to market value of traded positions only result in transfer of market value, and consequently collateral, among members.

Using the definition of κ_i given in Eq. (1.9), we can express both L_i and M_i in terms of the total asset value A_i. This yields a system of I equations, referred to as the *hedge equations*, given by:

$$\rho^2\theta^2(1-\kappa_i(t_m))^2 A_i(t_m) = \sigma^2\kappa_i(t_m)\sum_{j\neq i}\kappa_j(t_m)A_j(t_m),\ i = 1, 2, \ldots, I. \tag{1.12}$$

We next define an equilibrium, specific to our context, as a situation in which no member has incentive to change its trade ratio.

Definition 1.1. $\boldsymbol{\kappa}(t_m) := (\kappa_1(t_1), \kappa_2(t_m), \ldots, \kappa_I(t_m))' \in [0,1]^I$ *is an equilibrium profile at time t_m if the hedge equations given by* (1.12) *are simultaneously satisfied.*

It is clear that our equilibrium can be interpreted as a Nash Equilibrium. Interestingly, when each member seeks to hedge his undesired risk, there is only one set of trade ratios for which no member has an incentive to deviate from its chosen ratio.

Theorem 1. *Assume $A_i(t_m) > 0$ for all $i = 1, \ldots, I$. Then there exists a unique equilibrium profile at time t_m.*

We denote the vector of equilibrium trade ratios by $\boldsymbol{\kappa}^*$, and remark that, from Eq. (1.12), $\boldsymbol{\kappa}^*$ only depends on the relative fraction of assets that each member holds, $\frac{A_i}{A}$, where $A := \sum_{j=1}^{I} A_j$.

Combining Eq. (1.5), (1.4), (1.6), (1.7), and (1.10), we obtain

$$\begin{aligned}
\Delta_m \mathbf{A} &= \boldsymbol{\phi}(t_m) \circ \mathbf{A}(t_m)\Delta t + \sigma\sqrt{\mathbf{M}(t_m) \circ (M(t_m)\mathbf{1} - \mathbf{M}(t_m))} \circ \mathbf{Z}^{\boldsymbol{\Sigma}}_{m+1} + \theta \mathbf{L}(t_m) \circ \mathbf{Z}^{\boldsymbol{\Xi}}_{m+1}, \\
&= \boldsymbol{\phi}(t_m) \circ \mathbf{A}(t_m)\Delta t + \theta\sqrt{1-\rho^2}(\mathbf{1} - \boldsymbol{\kappa}^*(t_m)) \circ \mathbf{A}(t_m) \circ \mathbf{Z}^{\boldsymbol{\Psi}}_{m+1},
\end{aligned}$$

where the second equality follows from the unique equilibrium profile established in Theorem 1.

1.2.3 The hedging equilibrium model

Letting $\Delta t \to 0$, we obtain the continuous time model

$$d\mathbf{A} = \boldsymbol{\phi}\mathbf{A}dt + \theta \mathbf{L} \circ d\mathbf{W}^{\boldsymbol{\Xi}} + \sigma\sqrt{\mathbf{M} \circ (M\mathbf{1} - \mathbf{M})} \circ d\mathbf{W}^{\boldsymbol{\Sigma}}, \tag{1.13}$$

$$= \boldsymbol{\phi}\mathbf{A}dt + \theta\sqrt{1-\rho^2}(\mathbf{1} - \boldsymbol{\kappa}^*) \circ \mathbf{A} \circ d\mathbf{W}^{\boldsymbol{\Psi}}, \tag{1.14}$$

where $\mathbf{W}^{\boldsymbol{\Xi}}$, $\mathbf{W}^{\boldsymbol{\Sigma}}$, and $\mathbf{W}^{\boldsymbol{\Psi}}$ are I-dimensional Brownian motions with instantaneous correlation matrices $\boldsymbol{\Xi}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\Psi}$, respectively. For notational brevity, we suppress the time and state dependence throughout the paper wherever no confusion arises. The market clearing condition given in Eq. (1.11) now becomes

$$\sum_{i=1}^{I} \sqrt{M_i(M - M_i)} dW_i^{\boldsymbol{\Sigma}} = 0, \tag{1.15}$$

as shown in Proposition A.1.

To summarize, each member identifies its undesired risk exposures and constructs a trade account that can perfectly hedge them, taking into consideration the strategy followed by other members in choosing their trade account values. This leads to a dynamic equilibrium model for the total asset values of members. The above dynamics indicate that changes in asset values are purely driven by desired risk factors since members continuously hedge exposures to undesired risk factors through centralized trading.

2 Market concentration and systemic risk

This section studies market concentration effects arising in our dynamic equilibrium model of asset values. We quantify the dynamical properties of market concentration in section 2.1. We analyze how concentration propagates from trade accounts to members' total asset values in section 2.2. We discuss the ameliorating effects of capital raising and the ensuing systemic implications in section 2.3.

2.1 Market concentration: measure and dynamics

We use the Herfindahl index to measure market concentration. This is formally defined as

$$\eta := \frac{\sum_{i=1}^{I} A_i^2}{A^2},$$

where $A := \sum_{i=1}^{I} A_i$ denotes the aggregate asset value in the market. High values of the index are indicative of a more concentrated market with higher heterogeneity in size of the members. The index is bounded below by $\frac{1}{I}$ and above by 1. It equals $\frac{1}{I}$ when all members have equal size and 1 when all but one member have zero asset value.

In order to isolate the influence of trading on market concentration from the impact created by members' external capital injections, we consider the case of homogeneous capital raising rates. In this case, capital raising has no effect on market concentration. We decompose the desired risk of each member into a systematic and an idiosyncratic risk component. We denote by ψ_i the exposure of member i to systematic risk. This implies that the correlation between two members' desired risk profiles is given by $\psi_i\psi_j$. Under these circumstances, the next proposition shows that market concentration has an inherent tendency to increase.

Proposition 2.1. *Assume the dynamics given by Eq.* (1.14) *and that* $\boldsymbol{\phi} = \phi\mathbf{1}$*. Suppose* $\Psi_{i,j} = \psi_i\psi_j$ *for* $i \neq j$*, and* $0 \leq \psi_i^2 \leq \frac{1}{\sqrt{2}}$ *for each* i*. Then* $\frac{A_i^2}{A^2}$ *is a strict* $\mathbb{Q}-$*submartingale. Moreover,* η *is also a strict* $\mathbb{Q}-$*submartingale.*

We expect banks to have their desired exposures to market factors positively correlated. However, it is unlikely that the strength of the correlation exceeds values on the order of $\frac{1}{\sqrt{2}} \approx 71\%$. These high values are usually observed during periods of market distress, when asset classes become highly correlated (Hull (2012)). Hence, during normal market conditions, the submartingale property of the Herfindahl index is reflective of a market in which few members tend to dominate in size. This raises immediate concerns on systemic risk since failures of large banking entities tend to be more disruptive to the market (Acharya et al. (2010b) and Brownlees and Engle (2015)).

We remark that our result holds under the risk-neutral measure $\mathbb{Q}$, which can be interpreted as the product of the real world probability measure and marginal utility of a representative investor. The statement of η being a submartingale can thus be understood as either (i) the real world probability of the Herfindahl index increasing is high, or (ii) increases in the index are positively correlated with the occurrence of "bad" states of the world. The regulator would be concerned in both cases: in the first case for the likely increase in systemic risk; in the second case because systemic risk increases exactly when the market is performing poorly.

Our results also indicate that if business operations (desired risk factors) of members are diverse (correlations are low) then the market evolves towards a state of high heterogeneity in asset values. All else equal, the aggregate value of members' assets is less volatile when correlations are low, since the idiosyncratic risk components can be diversified away on the aggregate level. However, the distributional consequences are significant. As time progresses, asset values of some members will become significantly larger than others. This points to a trade-off between stable growth of aggregate asset value in the market and the ensuing market concentration consequences.

2.2 Propagation of concentration

In our model, the source of market concentration is the members' trade accounts. Indeed, in appendix A we show that in the absence of reallocation, the variation margin buffer dynamics converge to a Wright-Fisher diffusion process in the continuous time limit. A well known property of this process is *fixation*, where all but one component of the process reaches zero in a finite time horizon with probability one. That is, if we let the trading gains and losses in each period accumulate in the trade accounts, the variation margin buffers move towards a state of absolute concentration. We refer to this as the concentration effect of trading. See appendix A for the mathematical details.

We separately discuss the two main mechanisms which contribute to origination and propagation of market concentration. First, the trade account assets can be thought of as hoarded assets and allow for members with large trade account values to maintain high asset values (section 2.2.1). This allows the existing levels of market concentration to be preserved over time. Second, trade ratios depend monotonically on asset values (section 2.2.2 and 2.2.3), so that members allocate larger amounts of assets to the trade account when their asset values increase. This allows us to see how the concentration effect emerge on the level of members' total asset values. Taken together, these two mechanisms explain why market concentration tends to rise over time.

2.2.1 Market shocks and operational shocks

Market shocks occurring to the trade account value of a member do not impact the total value of its assets. To see this, suppose that there is a positive shock to the trade account value of member i. Because undesired risk is fully hedged, this shock is offset by a negative shock of the same magnitude to member i's loan book. Since trading is a zero-sum game, the gain of member i must correspond to collective losses experienced by the trade account values of other members. These losses, again, are compensated by commensurate gains in their loan book values. Hence, assets held in the trade account can be thought of as hoarded assets.

In addition, if member i's undesired risk were independent of other members' desired risk, the market shock would have no effect on the total asset of each member $j \neq i$, and consequently no effect on their equilibrium trade ratios. Concretely, member i would withdraw its gains from its trade account. The other members would instead extract capital from their loan books and inject it into their trade accounts so to hedge future undesired price movements in their loan books. Thus, market shocks can only indirectly impact the allocation of a member if they are correlated with operational shocks of other members.

As opposed to market shocks, operational shocks impact both total asset values and equilibrium allocations. We illustrate this by considering the situation where the total asset value of member i is large relative to the aggregate asset value in the market. In this case, its trade ratio approaches unity. To see this, set

$$\delta := \frac{\sigma^2}{\theta^2 \rho^2} > 0. \tag{2.1}$$

and rewrite Eq. (1.12) as

$$0 \le \frac{(1-\kappa_i^*)^2}{\kappa_i^*} = \delta \sum_{j \neq i} \kappa_j^* \frac{A_j}{A_i} \le \delta \sum_{j \neq i} \frac{A_j}{A_i} \to 0, \tag{2.2}$$

if $A_i \to \infty$. Thus, $\kappa_i^* \to 1$. Moreover, from Eq. (1.12) $\kappa_i^* \to 1$ implies that $\kappa_j^* \to 0$ for $j \neq i$. As a result, a negative shock occurring to the loan book of member i would have a small impact on its total asset value, since its balance sheet mostly consists of assets from its trade account. By contrast, the same shock (in percentage terms) has a substantial effect on the asset values of the remaining members, since their balance sheets are largely composed of assets belonging to their loan books.

The discussion above indicates that, at any point in time, the trade account value is a guaranteed minimum of asset value that the member will hold in the next period. The assets in their trade accounts are safe, and are thus hoarded.

2.2.2 Trade ratios and asset values: cross-section dependence

We first quantify the cross-section monotonic dependence of trade ratios on total asset value:

Proposition 2.2. *The following statements hold:*

- $\kappa_i^* \geq \kappa_j^*$ *if and only if* $A_i \geq A_j$.
- *If* $A_i > A_j$, *then* $\frac{M_i - M_j}{A_i - A_j} > \min\{\kappa_i^*, \kappa_j^*\}$.

The first statement of the proposition explains how market concentration is preserved: plotting members' asset values on the x-axis and equilibrium trade account values on the y-axis outlines a *superlinear* function. Since the trade account value enters concavely into the volatility structure ($\sigma\sqrt{\mathbf{M} \circ (M\mathbf{1} - \mathbf{M})}$), each incremental unit of capital allocated to hedging generates less volatility of the trade account value. Members with high asset values thus have even higher trade account allocations. Recall that from our previous discussion, shocks are most punitive to members whose assets are mainly invested in their loan books. A large member will thus have a tendency to remain large.

To analyze the impact of a change in members' asset values on the equilibrium trade ratios, and exclude the effects from changes in aggregate trade account value, we consider what would happen if the asset values of members 1 and 2 were to be switched, all else being equal. After the switch, the equilibrium trade account values remains the same, except for having the trade ratios of the two members switched. The second statement of proposition 2.2 shows, in this case, how concentration propagates: the member who has realized a gain in this switch invests a fraction of its gains higher than its original trade ratio. An increased fraction of its total assets is used to hedge undesired risk, hence a higher trade ratio.

2.2.3 Trade ratios and asset values: sensitivity analysis

This section studies the dependence of trade ratios on changes in the asset value of one member. We consider the specific case of two members. The main insight is that although equilibrium trade account values are monotonically increasing in members' asset values, the aggregate trade account value is not. It may increase or decrease depending on asset value heterogeneity.

Proposition 2.3. *Suppose* $I = 2$. *If* $\delta \neq 1$, *the system of equations* (1.12) *admits an explicit solution given by*

$$\begin{aligned} \kappa_1^*(A_1, A_2, \delta) &= \frac{\delta - \delta\frac{A_2}{A_1} - 2 + \sqrt{\left(\delta - \delta\frac{A_2}{A_1} - 2\right)^2 + 4(\delta - 1)}}{2(\delta - 1)}, \\ \kappa_2^*(A_1, A_2, \delta) &= \frac{\delta - \delta\frac{A_1}{A_2} - 2 + \sqrt{\left(\delta - \delta\frac{A_1}{A_2} - 2\right)^2 + 4(\delta - 1)}}{2(\delta - 1)}. \end{aligned} \tag{2.3}$$

If $\delta = 1$, *we have*

$$\kappa_1^*(A_1, A_2, 1) = \frac{A_1}{A_1 + A_2} = \lim_{\delta \to 1} \kappa_1^*(A_1, A_2, \delta), \tag{2.4}$$

$$\kappa_2^*(A_1, A_2, 1) = \frac{A_2}{A_1 + A_2} = \lim_{\delta \to 1} \kappa_2^*(A_1, A_2, \delta). \tag{2.5}$$

Moreover, κ_1^* *is increasing in* $\frac{A_1}{A_2}$, *while* κ_2^* *is decreasing in* $\frac{A_1}{A_2}$.

Using the expressions in proposition 2.3 we can compute the sensitivity of the trade account value of each member to its or to the other member's asset value. Recalling that $M_1^* = \kappa_1^* A_1$, and using that both κ_1^* and κ_2^* only depend on the ratio $\frac{A_1}{A_2}$, we obtain from the chain rule that

$$\frac{\partial M_1^*}{\partial A_1} = \frac{\partial \kappa_1^*}{\partial \frac{A_1}{A_2}} \frac{A_1}{A_2} + \kappa_1^*.$$

The fact that $\frac{\partial \kappa_1^*}{\partial \frac{A_1}{A_2}} > 0$ implies that M_1^* is a superlinear function of A_1. Therefore, when the asset value of a member doubles, the value of its trade account will at least double. Moreover,

$$\frac{\partial M_2^*}{\partial A_1} = -\frac{\partial \kappa_2^*}{\partial \frac{A_2}{A_1}} \left(\frac{A_2}{A_1}\right)^2 < 0.$$

Hence, when the asset value A_1 of member 1 increases, the trade account M_2^* of member 2 decreases. Notice that, from Eq. (1.8) and (1.10), a member can use the same amount of committed capital to hedge against more undesired price movements if the volatility of its trade account is high. Thus, hedging is less costly (in terms of committed capital) in this case. This explains why member 1 needs to more than double the value of its trade account. Since its counterparty is trading less, member 1 will have to trade more to obtain a position with volatility high enough to hedge its undesired price movements. Thus, we see that an increase in asset value (i) increases the amount hoarded and (ii) crowds out hedging activity of other members. Both of these effects lead to the propagation of concentration.

While collateral demand of each member is monotonic in its total asset value, the aggregate collateral demand is not. Indeed, using the expressions for κ_1^* and κ_2^* given in proposition 2.3, we can calculate the aggregate trade account value (collateral demand) as

$$M^* = M_1^* + M_2^* = \frac{-A_1 - A_2 + \sqrt{\delta^2(A_1 + A_2)^2 - 4\delta(\delta - 1)A_1A_2}}{\delta - 1},$$

and

$$\frac{\partial M^*}{A_1} = \frac{1}{\delta - 1}\left(-1 + \frac{(A_1 + A_2)\delta^2 - 2\delta(\delta - 1)A_2}{\sqrt{\delta^2(A_1 + A_2)^2 - 4\delta(\delta - 1)A_1A_2}}\right).$$

Consider the case of $\delta = 2$ and $A_1 = A_2$. Then

$$\frac{\partial M^*}{A_1} = -1 + \sqrt{2} > 0.$$

Thus when members have equal size, the superlinearity of trade account values dominates, and an increase in the asset value of a member leads to an increase in the aggregate trade account value. Next, consider the case when $\delta = 2$ and $A_2 = 2A_1$. Then

$$\frac{\partial M^*}{A_1} = -1 + \frac{2}{\sqrt{5}} < 0.$$

Hence, when members are heterogeneous in their sizes, the growth in asset values of the smallest member can lead to a decrease in the aggregate trade account value.

The above analysis shows that the impact of regulatory policies regarding asset transfers may have unintended consequences on market collateral demand. When regulators perform asset transfers, say, in the form of a bailout, to rescue a member in financial distress, an increase or decrease in market collateral demand can be generated. Increases in collateral demand are undesirable since collateral is most likely a scarce resource when the market is in overall distress. On the other hand, asset transfers may decrease collateral demand and lead to an excessive drop in prices of securities used as collateral. Our model can provide an analytical assessment of the impact of asset transfers on collateral demand.

2.3 Trade-off between market concentration and leverage

This section analyzes the impact on market concentration created when members raise capital in reaction to asset value losses. We consider the situation where members raise capital to preserve their market power, quantified by $\frac{A_i^2}{A^2}$, i.e. member i has high market power when this ratio is large. Since $\frac{A_i^2}{A^2}$ is a strict submartingale, the market power of each member grows, on average, over time. Consequently, market participants would have little incentive to alter the rising trend of the Herfindahl index ex ante. However, we may expect that a member who has realized losses during the last period chooses to raise capital at a faster rate so to increase the value of its total assets and compete with the other members.

We analyze a stylized setting where we can quantify the effects of heterogeneous capital raising rates on market concentration. Assume that all members raise the same amount of capital U per unit time. This means that small members raise capital at a higher rate than large members. Then we have the following:

Proposition 2.4. *Assume the dynamics given by Eq. (1.14). For $i = 1, \dots, I$, let $\phi A_i = U$, where U is an $\mathcal{F}_t$ adapted stochastic process. Then the drift coefficient of η is a monotonically decreasing function of U. Moreover, suppose that $\eta(t_0) > \frac{1}{I}$ at a fixed time t_0. Then there exists an $\mathcal{F}_t$ adapted stochastic process U^* such that*

$$\lim_{h \to 0} \frac{\mathbb{E}^{\mathbb{Q}}\left[\eta(t_0 + h) - \eta(t_0) | \mathcal{F}_{t_0}\right]}{h} < 0. \tag{2.6}$$

If small members raise capital at a faster rate than large members, the expected growth rate of market concentration would be reduced and can even become negative. The larger $U(t_0)$, the stronger the downward impact on expected growth rate at time t_0. This happens because when $U(t_0)$ is large, the inter-period change in loan book values become negligible relative to the high amounts of raised capital.

However, the regulator may not want members to raise capital incessantly. As pointed out by Adrian and Shin (2011), equity is sticky. Empirically observed equity book values of banks are aptly described by a constant growth rate. In addition, funding needs are mostly satisfied by debt and disciplined by market permitted leverage (Adrian et al. (2012)). Therefore, increasing capital raising rates most likely leads to an increase in the leverage ratio, another important measure of systemic risk (Bisias et al. (2012)).

3 Preventive policies

We have seen in the previous section that market concentration can be reduced if members are allowed to freely raise capital. This points to a trade-off the regulator may face when controlling two important systemic risk measures. If the regulator freely allows members to raise capital, their leverage ratios may increase. If the regulator places strong restrictions on the capital levels which members are allowed to raise, market concentration would tend to increase. Section 3.1 proposes a policy imposing systemic risk charges to members along with a trade ratio mandate, with the objective of helping the regulator address the trade-off. Section 3.2 analyzes potential reactions of members to such a charge mechanism in case no trade ratio mandate were to be imposed.

3.1 Systemic risk charge based policy

The failure of large financial entities may create significant negative externalities. The two-step policy described next forces members to internalize the size externalities arising in our model.

1. Mandate members to match their trade ratios to the equilibrium value $\boldsymbol{\kappa}^*$ obtained by solving the system of equations (1.12).
2. Concurrently and for each clearing member, impose a charge at rate μ per unit time on the trade account value and transfer to each a fixed percentage, ψ, of the aggregate trade account value.

When this policy is in place, the modified dynamics of the trade account values is given by

$$\Delta_n \mathbf{M} = -\mu \mathbf{M} \Delta t + \psi M \mathbf{1} \Delta t + \sigma \sqrt{\mathbf{M} \circ (M\mathbf{1} - \mathbf{M})} \mathbf{Z}_n^{\boldsymbol{\Sigma}}. \tag{3.1}$$

We also impose an instantaneous budget constraint

$$\sum_{i=1}^{I} \mu M_i = \sum_{i=1}^{I} \psi M, \tag{3.2}$$

so that aggregate net transfers from the regulator are zero at any point in time, hence making charges self-funding. Notice that the instantaneous budget constraint implies $\mu = I\psi$. We refer to μ as the *policy rate.*

The continuous time asset value dynamics of the members, adjusted for systemic risk charges, are then given by:

$$d\mathbf{A} = \boldsymbol{\phi} \mathbf{A} dt - \mu \left(\mathbf{M} - \frac{M}{I} \mathbf{1} \right) dt + \sigma \sqrt{\mathbf{M} \circ (M\mathbf{1} - \mathbf{M})} \circ d\mathbf{W}^{\boldsymbol{\Sigma}} + \theta \mathbf{L} \circ d\mathbf{W}^{\boldsymbol{\Xi}} \tag{3.3}$$

$$= \boldsymbol{\phi} \mathbf{A} dt - \mu \left(\mathbf{M} - \frac{M}{I} \mathbf{1} \right) dt + \theta \sqrt{1-\rho^2} (\mathbf{1} - \boldsymbol{\kappa}^*) \circ \mathbf{A} \circ d\mathbf{W}^{\boldsymbol{\Psi}}, \tag{3.4}$$

where we recall that the trade allocation ratio $\boldsymbol{\kappa}^*$ is mandated by the regulator. In other words, members would need to follow the same hedging behavior as in the absence of any policy.

The next proposition shows that the policy rate parameter μ can be suitably used by the regulator to control market concentration. If the regulator deems it to be too high, it can choose higher values of μ and reduce the average growth rate of market concentration.

Proposition 3.1. *Assume the dynamics given by Eq.* (3.4)*, where* $\boldsymbol{\phi} = \phi \mathbf{1}$*. Then the drift coefficient of* η *is a monotonically decreasing function of* μ*. Moreover, let* t_0 *be a fixed time such that* $\eta(t_0) > \frac{1}{I}$*. Then there exists a* $\mathcal{F}_t$ *adapted stochastic process* μ *such that*

$$\lim_{h \to 0} \frac{\mathbb{E}^{\mathbb{Q}} \left[\eta(t_0 + h) - \eta(t_0) | \mathcal{F}_{t_0} \right]}{h} < 0. \tag{3.5}$$

The purpose of the systemic charges is two-fold. From a systemic risk standpoint, the objective is to control the evolution of market concentration and avoid too-big-to-fail scenarios. From a macroprudential perspective, controlling market concentration may mitigate credit contraction when negative shocks occur. In light of proposition 2.2, in a highly concentrated financial network small members allocate a high proportion of their assets to their loan books. As a consequence, they may contract their credit provisions more drastically when they experience a negative shock to their loan books.

Since the charge allocation policy is self-funding, market concentration can be kept low without lightening regulatory constraints on external capital raising levels through debt issuance. The systemic risk charge is essentially a charge on size. In this respect, it presents similarities with the policy proposed by Acharya et al. (2010b), in which financial entities are taxed based on the extent and likelihood of their contribution to systemic risk. In a related study, Acharya et al. (2010a) describe how externalities can be internalized by institutions via the imposition of a tax on systemic expected shortfall (SES). Moreover, they show that regressing SES on institution size gives a positive coefficient that is both statistically and economically significant. Thus, our proposed systemic risk charges policy supports the taxing system proposed by Acharya et al. (2010a). While Acharya et al. (2010a) apply taxes on the basis of the expected losses incurred by the financial system, conditional on the occurrence of a systemic crisis, and propose to reward firms carrying less risk, our systemic risk charge policy rewards members with smaller sizes to mitigate market concentration.

3.2 The regulator's trade ratio mandate

We remark that the hedge equations given by (1.12) would not hold if the regulator specifies only a policy rate, but does not impose the trade ratio mandate to members. When only a systemic risk charge is imposed, members would most likely respond with a change in their hedging behavior.

One potential response is that all members could uniformly hedge smaller portions of undesired risk with respect to what is considered optimal by the model. Since members with low trade account values receive higher net transfers from the regulator, there may be a race to exit if members deem systemic risk charges to be too high. This may be undesirable from the point of view of the clearinghouse, which would see its profits decrease as a result of the reduced volume of trading activities conducted with the members. Moreover, from the regulator's point of view, it would be governing a system where members are exposed to higher amount of undesired risk.

Another potential response from the members is that they may cease to act in a competitive nature. A necessary assumption made in the derivation of the trade account value dynamics of members is that there are no arbitrage opportunities when members establish cleared trades. Under a systemic risk charge, members may choose suboptimal hedges so that overall benefits from trading with the clearinghouse are optimal. In this case, our derived dynamics may not serve as a good description of trading activities.

Our policy mechanism prevents both of the above responses. Since systemic risk charges to members are fixed given their trade ratio requirements, it is optimal for them to choose a portfolio composition that hedges all of their undesired risk. Clearing members thus have no reason to grant arbitrage trading opportunities to others and our trade account dynamics remain a good description of gains and losses.

4 Testable predictions

Our study provides several testable predictions for a centrally cleared trading network. In particular, our analysis suggests a line of empirical research that tests relations between the hedging strategies of market participants, their business operations, and market collateral demand.

First, our model predicts a superlinear relationship between asset value and the amount of capital committed to trading. The results in propositions 2.2 and 2.3 indicate that hedging is increasingly costly. Each incremental unit of capital committed to hedging is less efficient, and thus a member must allocate an increasingly larger fraction of its assets to hedging if its asset value increases. Empirically, this may be tested using cross-sectional regressions estimating the relation between enterprise value and collateral demand of institutions with similar business profiles. The collateral demand may be estimated using an approach similar to the one proposed by Duffie et al. (2015). Alternatively, a time series approach can be employed where incremental change of asset value is regressed against change in collateral demand.

Second, our model indicates that volatility of the member's trade account is not only governed by its decisions, but also by the decisions of all other members. As expected, member i's collateral demand, designed to cover variations in the market value of its portfolio, is a good predictor of its realized volatility. Interestingly, the collateral demand of all clearing members excluding i, is also a good predictor of member i's realized volatility.

Third, market shocks have a much smaller effect on asset allocation decisions than operational shocks of the same magnitude. Market shocks occurring to member i's trade account are naturally hedged by undesired movements in the loan book portfolio, thus have no direct effect on its asset values and allocations (section 2.2). Such shocks can only affect its allocations indirectly by impacting other members' asset values, which in turn affect member i''s allocation in equilibrium. The hedge equations in Eq. (1.12) can be used to test such a relation between change in asset value and allocations. Another way of testing this relation is identifying pure market shocks and operational shocks and performing an impulse response analysis on the asset allocations and trade ratios. We expect operational shocks to have a longer lasting effect on collateral demand than market shocks.

Fourth, a fundamental regulatory concern is the understanding and monitoring of the dynamics of market concentration. Our analysis shows that there may exist a trade-off between controlling market concentration and leverage (proposition 2.4). Our model predicts that in a fully centrally cleared network market concentration has an inherent tendency to increase, but this effect can be hidden if members raise capital at heterogeneous rates. Hence, empirical work can be conducted to separate the effects of capital raising from trading. One direct approach would be to identify historical periods where capital raising rates were relatively homogeneous across members and estimate changes in market concentration during this period.

Fifth, while diverse business operations of financial institutions (diversity in desired risks) allows for more stable growth in the aggregate asset value, it also creates undesirable distributional consequences (proposition 2.1). While idiosyncratic risks are diversified away on the aggregate level, they are the driving force behind heterogeneity in members' asset values, and may thus lead to a concentrated financial network. One way to analyze concentration effects is to identify different periods, each associated with a specific level of asset returns correlation, and separately estimate the distributional changes in asset values.

5 Concluding remarks

We provide a theoretical framework in which the implementation of two important financial reforms, namely the centralized clearing of trades and the restriction on proprietary trading, is taken into account. We show the uniqueness of an intra-temporal equilibrium allocation profile, or equivalently stated, the dynamic optimal strategies employed by clearing members as they hedge their loan books. We then analyze the asset value dynamics of clearing members arising from the equilibrium path. We find that:

1. Capital costs of members' hedging positions depend on those of all other members. The amount of collateral required to hedge per unit undesired risk is higher for a member when other members have larger trade accounts.

2. Hedging is increasingly costly. All else equal, doubling the asset value of a member would more than double its equilibrium trade ratio.

3. Market concentration tends to rise. When risk factors driving loan books are diverse, market concentration increases and persists as the large members further increase in size. Hedging, while risk-mitigating on an individual level, contributes to the emergence of size externalities on the systemic level. We develop a systemic risk charge mechanism to let members internalize these externalities.

The systemic risk charge is a self-financing charge, enforced by the CCP and directed by the regulator, to limit concentration risk. It consists of two parts. The first is a fee, different for each member, which is proportional to the size of the member's margin trading account at the CCP. The second is a rebate, the same for each member, and which relates to the aggregate margin trading account across clearing members at the

CCP. Clearing members pay (or receive) to the extent the fee is greater (or lesser) than the rebate. Large clearing members are expected to be net financiers of the systemic risk charge, while smaller clearing members are expected to be net recipients. The regulator controls the fee through a systemic risk charge policy rate, which determines to what extent and how quickly concentrations across clearing members may reduce.

Our model provides testable predictions regarding the relation between hedging strategies of market participants, their business operations, and market collateral demand. These relations can assist regulators to detect structural changes in members' behavior and to analyze the impact of preventive policies aiming at financial stability. For example, our framework helps the regulator assess whether a certain member is being too aggressive in risk-taking compared to its peers. This can be achieved by contrasting the member's realized trade ratio to our model's equilibrium trade ratio.

Our model identifies important risks regarding the growth of central clearing. While financial regulatory reform has promoted central clearing, in the course of this development, CCPs may become exposed to too-big-to-fail clearing members. The cost of externalities introduced under central clearing could exceed the hedging benefits associated with it.

While the model put forward in this paper analyzes systemic risk dynamics under central clearing, it can also be embedded in a structural credit risk framework to analyze default risk of financial institutions. The continuous time asset value process implied by our model can be used to obtain dynamics of default probabilities, and aid the establishment of dynamic capital requirements. Additionally, our framework can be extended to examine systemic implications under different behavioral trading assumptions. For instance, when retail investors trade on a futures exchange for speculation rather than hedging risk, the equilibrium allocation would no longer be determined by the hedge equations. Nevertheless, the component of our model describing profits and losses from centrally cleared trading is invariant to the specific trading purpose, and can be used in conjunction with a variety of trading behavioral models. We leave the construction of such a comprehensive framework for future research.

A The concentration effect from trading

In this appendix we show that (i) when reallocation is not allowed, the variation margin buffer dynamics converge to the $I-$allele neutral Wright-Fisher diffusion process, and that (ii) the volatility structure of the trade account value dynamics given in Eq. (1.3) is naturally associated to that of the Wright-Fisher diffusion process.

We consider the case where members do not choose their allocations due to hedging needs. Rather, they choose allocations so that their new variation margin buffers are the same as the residual variation margin buffers after accounting for variation margin payments. This "hands-off" case allows us to isolate the effects of trading, as members do not actively change capital allocations due to losses. Using the same notation as in section 1.1.2, and setting $t = k\Delta t$, $k \in \mathbb{N}$, we have:

$$V_i((k+1)\Delta t) = V_i(k\Delta t) + R(k\Delta t + \Delta t).$$

Notice that for all k, the aggregate trade account value $V(k\Delta t) = V(0)$, since there is no withdrawal or infusion of capital into trading. Consider the normalized variation margin buffer processes $v_i^{(N)}(k\Delta t) := V(0)^{-1} V_i(k\Delta t)$. We then have that $\sum_{i=1}^{I} v_i^{(N)}(k\Delta t) = 1$ for all k.

$$v_i^{(N)}((k+1)\Delta t) = \frac{V_i((k+1)\Delta t)}{V(0)} = \frac{N_i((k+1)\Delta t) \times V(0)/N}{V(0)} = \frac{N_i(k\Delta t + \Delta t)}{N}. \quad \text{(A.1)}$$

Notice that Eq. (1.2) and Eq. (A.1) together show that $\mathbf{Y} := N \times (v_1^{(N)}, \ldots, v_I^{(N)})$ follows an $I-$allele Wright-Fisher Markov Chain model (Feller et al. (1951)), which has a well known diffusive limit: $(2N)^{-1}\mathbf{Y}(\lfloor 2Nt \rfloor) \xrightarrow{d} \mathbf{p}(t)$, where $\mathbf{p}(t)$ is referred to as the I-allele Wright-Fisher diffusion process (Guess (1973)). For a precise definition of the process, see theorem A.2 and its associated proof.

An important feature of this process is fixation, defined as the stopping time

$$\tau := \min_t \{p_i(t) = 0, \text{for all but one } i = 1, \ldots, I\}.$$

It is well known that $\mathbb{E}[\tau | \mathbf{p}(0) = \mathbf{p}_0] < \infty$ (Guess (1973)). That is, the process moves towards a state of absolute concentration where $\eta(\mathbf{V}) = 1$. When reallocation is prohibited, the trade accounts become highly concentrated over time. This is the "concentration effect" of trading.

Next, we show that the volatility structure of the trade account value dynamics coincides with the one of the Wright-Fisher diffusion process. Let $\Delta_{I-1} := \{x \in \mathbb{R}^{I-1} | x_i \geq 0, i = 1 \ldots I-1, \sum_{i=1}^{I-1} x_i \leq 1\}$ and $\delta_{i,j}$ be the Kronecker delta function. Let $x = (x_1, \ldots, x_I) \in \mathbb{R}_+^I$, where the subscript $+$ denotes the nonnegative orthant. Denote $\tilde{x} := \max_i x_i$ and $\bar{x} := \sum_{i=1}^{I} x_i$. Notice that $\tilde{x} \leq \bar{x}$, with equality if and only if $I-1$ components of x are zero. Define the matrix-valued function $\mathbf{\Sigma}(x) : \mathbb{R}_+^I \to \mathbb{R}^I \times \mathbb{R}^I$ as:

$$\mathbf{\Sigma}_{i,j}(x) = \begin{cases} \delta_{i,j} - (1-\delta_{i,j})\sqrt{\frac{x_i}{\bar{x}-x_i}\frac{x_j}{\bar{x}-x_j}}, \ \tilde{x} < \bar{x} \\ 0, \ \tilde{x} = \bar{x} \end{cases}. \quad \text{(A.2)}$$

One can easily show that if $\mathbf{G}$ is a random variable following a multinomial distribution with parameters $(n, \frac{\mathbf{V}}{V})$, then $\mathbf{\Sigma}(\mathbf{V})$ is the correlation matrix of $\mathbf{G}$. Although the I-allele Wright-Fisher diffusion has been well studied, to the best of our knowledge it has always been characterized in terms of its infinitesimal generator. Since we need the SDE representation in our discussion, we present our own proof. We start with the following proposition showing that the market clearing conditions are satisfied when $\mathbf{\Sigma}$ is given by Eq. (A.2).

Proposition A.1. *For any nonnegative $I-$dimensional process* $\mathbf{X}$*, we have*

$$\sum_{i=1}^{I} \sqrt{X_i(t)(\bar{X}(t) - X_i(t))} dW_i^{\mathbf{\Sigma}} = 0 \tag{A.3}$$

The above proposition plays a crucial role in the proof of the SDE representation of the Wright-Fisher diffusion process:

Proposition A.2. *Let* $\mathbf{p}(t)$ *be a I-allele Wright-Fisher diffusion process with initial value* $\mathbf{p}(0) = \mathbf{p}_0$*. Then* $\mathbf{p}(t)$ *is the unique weak solution to the SDE*

$$\begin{cases} d\mathbf{X}(t) = \sqrt{\mathbf{X}(t) \circ (\mathbf{1} - \mathbf{X}(t))} \circ \mathbf{\Gamma}(\mathbf{X}(t)) d\mathbf{B}(t) \\ \mathbf{X}(0) = \mathbf{p}_0 \end{cases} . \tag{A.4}$$

Here $\mathbf{\Gamma}(x)$ *is the unique matrix such that* $\mathbf{\Gamma}^2(x) = \mathbf{\Sigma}(x)$*, and* $\mathbf{B}$ *is a* $I-$ *dimensional standard Brownian motion.*

We next give the proof for the discretized version of the market clearing condition, given in Eq. (1.11), and used in the construction of the equilibrium allocation profile in Section 1.2.2:

Proposition A.3. *For any nonnegative $I-$dimensional process* $\mathbf{X}$*, we have*

$$\sum_{i=1}^{I} \sqrt{X_i(t_m)(\bar{X}(t_m) - X_i(t_m))} Z_{i,n+1}^{\mathbf{\Sigma}} = 0 \tag{A.5}$$

All proofs of the above propositions are reported in appendix B.

Last, we discuss the role of $\sigma'^{-2} := N\Delta t$. σ' is essentially a time change parameter that allows us to scale the trade account value dynamics depending on the target time frame (hourly, daily, weekly..., etc). To see this, suppose that for some reference time frame t', we have $(2N)^{-1}\mathbf{Y}(\lfloor 2Nt' \rfloor) \xrightarrow{d} \mathbf{p}(t')$. The dynamics of $V_i(t)$ under the time change $t = \frac{1}{\sigma'^2}t'$ are obtained as follows. First

$$V_i(t) = p_i(t) V\left(\frac{t'}{\sigma'^2}\right) = p_i(t)V(0).$$

and thus, since $\sigma'\mathbf{B}(t)$ is a Brownian motion the original time t',

$$\begin{cases} d\mathbf{V}(t) = \sigma' \sqrt{\mathbf{V}(t) \circ (V\mathbf{1} - \mathbf{V}(t))} \circ \mathbf{\Gamma}(\mathbf{V}(t)) d\mathbf{B}(t) \\ \mathbf{V}(0) = \mathbf{V}_0 \end{cases} . \tag{A.6}$$

Notice that $\int_0^t \mathbf{\Gamma}(\mathbf{V}(s))d\mathbf{B}(s)$ is a correlated Brownian motion with instantaneous correlation matrix $\mathbf{\Sigma}(\mathbf{V}(t))$.

B Proofs

Proof of Theorem 1. For notational convenience, we define the quantity $\delta := \frac{\sigma^2}{\theta^2\rho^2} > 0$. In addition, since we are considering a fixed instant in time, we will suppress all time arguments t_m. Without loss of generality, we assume that the sequence of asset values is in decreasing order, i.e. $A_1 \geq A_2 \geq \cdots \geq A_I > 0$.

Notice that the hedge equations in (1.12) may be written in three equivalent forms: for $i = 1, \ldots, I$,

1.
$$(A_i - M_i)^2 = \delta M_i \sum_{j \neq i} M_j,$$
whose solution is denoted by $(\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_I) \in \prod_{i=1}^{I}[0, A_i]$.

2.
$$\begin{cases} (A_i - M_i)^2 = \delta M_i(M - M_i), \\ M = \sum_{i=1}^{I} M_j, \end{cases} \tag{B.1}$$
whose solution is denoted by $(M^*, M_1^*, M_2^*, \ldots, M_I^*) \in [0, A] \times \prod_{i=1}^{I}[0, A_i]$, where $A := \sum_{i=1}^{I} A_i$.

3.
$$(1 - \kappa_i)^2 A_i = \delta \kappa_i \sum_{j \neq i} \kappa_j A_j, \tag{B.2}$$
whose solution is denoted by $(\kappa_1^*, \kappa_2^*, \ldots, \kappa_I^*) \in [0, 1]^I$.

The second form follows from the first after introducing the slack variable M. The third form follows from the first form by using the relation $M_i = \kappa_i A_i$. Obviously, there is a one-to-one correspondence between solutions of each form.

We prove existence using the representation in (B.2) and then uniqueness using the representation in (B.1).

Proof of Existence. Let $\mathbf{x} := (x_1, x_2, \ldots, x_n)$ and define $U_i(\mathbf{x}) := \sum_{j \neq i} x_j A_j \geq 0$. Applying the quadratic formula to Eq. (B.2), we can solve for

$$\kappa_i = 1 + \frac{\delta U_i(\boldsymbol{\kappa})}{2A_i} \pm \sqrt{\left(1 + \frac{\delta U_i(\boldsymbol{\kappa})}{2A_i}\right)^2 - 1}.$$

In other words, if we define

$$\begin{aligned} \psi(\mathbf{x}) &:= (\psi_1(\mathbf{x}), \psi_2(\mathbf{x}), \ldots, \psi_n(\mathbf{x})), \\ \psi_i(\mathbf{x}) &:= 1 + \frac{\delta U_i(\mathbf{x})}{2A_i} - \sqrt{\left(1 + \frac{\delta U_i(\mathbf{x})}{2A_i}\right)^2 - 1}, \end{aligned} \tag{B.3}$$

then any $\mathbf{x}^* \in [0, 1]^I$ such that $\psi(\mathbf{x}^*) = \mathbf{x}^*$ must be a solution of Eq. (B.2). For any $\mathbf{x} \in [0, 1]^I$, $U_i \geq 0$, and obviously $\psi_i(\mathbf{x}) \geq 0$ from Eq. (B.3). We also have that $\psi(\mathbf{x}) \leq 1$ since we can rewrite Eq. (B.3) as

$$\psi_i(\mathbf{x}) = 1 + \frac{\delta U_i(\mathbf{x})}{2A_i} - \sqrt{\left(\frac{\delta U_i(\mathbf{x})}{2A_i}\right)^2 + \frac{\delta U_i(\mathbf{x})}{A_i}}.$$

Thus ϕ is a continuous mapping from $[0, 1]^I$ to $[0, 1]^I$, and by applying Brouwer's fixed-point theorem there exists $\mathbf{x}^* \in [0, 1]^I$ such that $\phi(\mathbf{x}^*) = \mathbf{x}^*$. ☐

Proof of Uniqueness. Since we have established the existence of a solution to Eq (B.2), there must also exist a solution to Eq (B.1). Here show that the solution of Eq. (B.1) is unique.

We first observe that, given a solution $\mathbf{M}^* := (M^*, M_1^*, M_2^*, \ldots, M_I^*)$ to the system of equations (B.1), it must be the case that either $M_i = f_i^-(M)$ or $M_i = f_i^+(M)$, where

$$f_i^-(y) := \frac{2A_i + \delta y - \sqrt{(2A_i + \delta y)^2 - 4A_i^2(\delta + 1)}}{2(1 + \delta)}, \tag{B.4}$$

$$f_i^+(y) := \frac{2A_i + \delta y + \sqrt{(2A_i + \delta y)^2 - 4A_i^2(\delta + 1)}}{2(1 + \delta)}. \tag{B.5}$$

This can be directly seen by solving the first equation in (B.1) using the quadratic formula.

Next, we define two classes of solutions. A solution $\mathbf{M}^*$ is of Class I if it has the functional form

$$(M^*, f_1^+(M^*), f_2^-(M^*), \ldots, f_I^-(M^*)),$$

while it is of Class II if it has the functional form

$$(M^*, f_1^-(M^*), f_2^-(M^*), \ldots, f_I^-(M^*)).$$

The proof of uniqueness consists of four steps:

Step 1. All feasible solutions must either be of Class I or Class II

In this step we prove three auxiliary lemmas, B.1–B.3. These are used to prove Lemma B.4, which shows that all feasible solutions must be either of Class I or Class II.

Lemma B.1. *It holds that for each $i = 1, \ldots, I$, $f_i^+(y)$ is strictly increasing in y, while $f_i^-(y)$ is strictly decreasing in y. Moreover,*

$$M^* \geq M^\dagger := \frac{2A_1}{1 + \sqrt{1 + \delta}}. \tag{B.6}$$

Proof of Lemma B.1. Since

$$\frac{\partial f_i^+(y)}{\partial y} = \frac{\delta}{2(1 + \delta)} \left(1 + \frac{2A_i + \delta y}{\sqrt{(2A_i + \delta y)^2 - 4A_i^2(1 + \delta)}}\right) > 0,$$

it follows that $f_i^+(y)$ is strictly increasing in y. Moreover,

$$\frac{\partial f_i^-(y)}{\partial y} = \frac{\delta}{2(1 + \delta)} \left(1 - \frac{2A_i + \delta y}{\sqrt{(2A_i + \delta y)^2 - 4A_i^2(1 + \delta)}}\right) < 0,$$

which implies that $f_i^-(y)$ is strictly decreasing in y. Since M_i^* is real for each i, the discriminant of the quadratic equation gives:

$$(2A_i + \delta M^*)^2 - 4A_i^2(\delta + 1) \geq 0,$$

which is equivalent to

$$M^* \geq \frac{2A_i}{1 + \sqrt{1 + \delta}}.$$

Since the above upper bound must hold for each $i = 1, \ldots, I$, we obtain

$$M^* \geq \max_i \left\{\frac{2A_i}{1 + \sqrt{1 + \delta}}\right\} = \frac{2A_1}{1 + \sqrt{1 + \delta}}.$$

□

Lemma B.2. *Suppose that $M_i^* = f_i^+(M^*)$. Then $M^* \leq A_i$.*

Proof of Lemma B.2. The proof goes by contradiction. Assume $M^* > A_i$. It can be immediately verified that $f_i^+(A_i) = A_i$. We know from Lemma B.1 that $f_i^+(y)$ is strictly increasing in y. Hence, $M_i^* = f_i^+(M^*) > f_i^+(A_i) = A_i$, leading to a contradiction since $M_i^* \in [0, A_i]$ by definition of a solution of Eq. (B.1). □

Lemma B.3. *$M_i^* \geq M_j^*$ if and only if $A_i \geq A_j$.*

Proof of Lemma B.3. First we show that $M_i^* \in (0, A_i)$. If $M_i^* = 0$, plugging this into Eq. (B.1) gives $A_i = 0$, a contradiction. If $M_i^* = A_i$, the same equation gives $M^* = A_i = M_i^*$, which implies $M_j^* = 0$ for $j \neq i$, leading again to a contradiction.

Since $A_i > M_i$, we can rewrite Eq. (B.1) as

$$A_i = M_i^* + \sqrt{\delta}\sqrt{-\left(\frac{M^*}{2} - M_i^*\right)^2 + \frac{M^{*2}}{4}}.$$

Define the function

$$a(y) := y + \sqrt{\delta}\sqrt{-\left(\frac{M^*}{2} - y\right)^2 + \frac{M^{*2}}{4}}.$$

Then it is obvious that $a(y)$ is strictly increasing on $(0, \frac{M^*}{2})$.

Suppose $M_i^* > M_j^*$. Then it must hold that $M_j \leq \frac{M^*}{2}$ otherwise $M^* \geq M_i^* + M_j^* > \frac{M^*}{2} + \frac{M^*}{2} = M^*$, which is a contradiction. Next, we consider two separate cases. Suppose first that $M_i^* \leq \frac{M^*}{2}$. Then $A_j = a(M_j^*) < a(M_i^*) = A_i$. Next, suppose that $M_i^* > \frac{M^*}{2}$. Then for all $j \neq i$, $M_j^* < M^* - M_i^* < \frac{M^*}{2} < M_i^*$. Thus

$$A_j = a(M_j^*) < a(M^* - M_i^*) = M^* - M_i^* + \sqrt{\delta}\sqrt{-\left(\frac{M^*}{2} - M^* + M_i^*\right)^2 + \frac{M^{*2}}{4}} < a(M_i^*) = A_i.$$

This concludes the proof of the "if" statement. The "only if" direction follows by notational symmetry and antisymmetry of the $\geq$ relation. □

Lemma B.4. *For $i \geq 2$, it holds that $M_i^* = f_i^-(M^*)$. Thus, $\mathbf{M}^*$ must be of either Class I or Class II.*

Proof of Lemma B.4. Suppose, by contradiction, that there exists $i \geq 2$ such that $M_i^* = f_i^+(M^*)$. From Lemma B.2, we have that $M^* \leq A_i$. Thus

$$M_i^* = f_i^+(M^*) = \frac{2A_i + \delta M^* + \sqrt{(2A_i + \delta M^*)^2 - 4A_i^2(\delta + 1)}}{2(1+\delta)} \geq \frac{2M^* + \delta M^*}{2(1+\delta)} > \frac{M^*}{2}. \quad \text{(B.7)}$$

By Lemma B.3, $M_1^* \geq M_i^*$, thus $M^* \geq M_i^* + M_1^* > \frac{M^*}{2} + \frac{M^*}{2} = M^*$, which is a contradiction. □

Step 2. There is at most one solution of Class II

Recall $M^\dagger$ defined in Eq. (B.6). We have the following

Lemma B.5. *Define the function $m^-(y) := -y + \sum_{i=1}^I f_i^-(y)$. There are no solutions of Class II if and only if $m^-(M^\dagger) < 0$. Moreover, there is exactly one solution of Class II if and only if $m^-(M^\dagger) \geq 0$.*

Proof of Lemma B.5. By Lemma B.1, we only need to consider values of M^* such that $M^\dagger \leq M^* \leq A$. Notice that $\mathbf{M}^*$ is a solution of class II if and only if $y = M^*$ is a solution of the equation

$$0 = -y + \sum_{i=1}^I f_i^-(y). \quad \text{(B.8)}$$

Hence we can focus on solutions to Eq. (B.8). We next show via a direct calculation that $m^-(A) < 0$, where we recall that $A = \sum_{i=1}^I A_i$ is the sum of members' asset values. Since for $x > z > 0$ it holds that $\sqrt{x} - \sqrt{x-z} < \sqrt{z}$, for any $y > 0$

$$f_i^-(y) = \frac{2A_i + \delta y - \sqrt{(2A_i + \delta y)^2 - 4A_i^2(\delta+1)}}{2(1+\delta)} < \frac{\sqrt{4A_i^2(1+\delta)}}{2(1+\delta)} < A_i. \tag{B.9}$$

Therefore, we have

$$m^-(A) = -A + \sum_{i=1}^I f_i^-(A) = \sum_{i=1}^I (f_i^-(A) - A_i) < 0.$$

In addition,

$$\frac{\partial m^-(y)}{\partial y} = -1 + \sum_{i=1}^I \frac{\partial f_i^-(y)}{\partial y} < 0$$

by Lemma B.1. Thus, the function on the right hand side of Eq. (B.8) has a strictly negative derivative everywhere.

This implies that if $m^-(M^\dagger) \geq 0$ there must exist exactly one solution of Eq. (B.8) in $[M^\dagger, A]$, since $m^-(M^\dagger)m^-(A) \leq 0$ and the derivative of $m^-(y)$ is strictly negative. Additionally, it implies that if $m^-(M^\dagger) < 0$ no solution can exist in $[M^\dagger, A]$ since $m^-(M^\dagger)m^-(A) > 0$. □

Step 3. There is at most one solution of Class I

Lemma B.6. *Define the function* $m^+(y) := -y + f_1^+(y) + \sum_{i=2}^I f_i^-(y)$. *There are no solutions of Class II if and only if* $m^+(M^\dagger) > 0$. *Moreover, there is exactly one solution of Class II if and only if* $m^+(M^\dagger) \leq 0$.

Proof of Lemma B.6. By Lemma B.1 and B.2, for the case of class I solutions we only need to consider values of M^* such that $M^\dagger \leq M^* \leq A_1$. Notice that $\mathbf{M}^*$ is a solution of class I if and only if $y = M^*$ is a solution of the equation

$$0 = -y + f_1^+(y) + \sum_{i=2}^I f_i^-(y). \tag{B.10}$$

Hence, we can restrict attention to solutions of Eq. (B.10). It can be directly verified that $m^+(y)$ is twice continuously differentiable on $(M^\dagger, \infty)$. The proof of the result will be based on an analysis of the second derivative of m^+. First, we calculate the following derivatives:

$$\frac{\partial^2 f_i^-(y)}{\partial y^2} = 2\delta^2 \frac{A_i^2}{\left(\sqrt{(2A_i + \delta y)^2 - 4A_i^2(1+\delta)}\right)^3}.$$

$$\frac{\partial^2 f_1^+(y)}{\partial y^2} = -2\delta^2 \frac{A_1^2}{\left(\sqrt{(2A_1 + \delta y)^2 - 4A_1^2(1+\delta)}\right)^3}.$$

$$\frac{\partial m^+(y)}{\partial y} = \frac{\delta}{2(1+\delta)} \left(1 + \frac{2A_1 + \delta y}{\sqrt{(2A_1 + \delta y)^2 - 4A_1^2(1+\delta)}} + \sum_{i=2}^I \left(1 - \frac{2A_i + \delta y}{\sqrt{(2A_i + \delta y)^2 - 4A_i^2(1+\delta)}}\right)\right). \tag{B.11}$$

$$\frac{\partial^2 m^+(y)}{\partial y^2} = -2\delta^2 \frac{A_1^2}{\left(\sqrt{(2A_1 + \delta y)^2 - 4A_1^2(1+\delta)}\right)^3} + 2\delta^2 \sum_{i=2}^I \frac{A_i^2}{\left(\sqrt{(2A_i + \delta y)^2 - 4A_i^2(1+\delta)}\right)^3}. \tag{B.12}$$

Next, we show that:

(P1) $\lim_{y \to M^\dagger} \frac{\partial^2 m^+(y)}{\partial y^2} = -\infty$.
To see this, we first show that $A_1 > A_2$. Suppose, by contradiction, that this is not the case, i.e. $A_1 = A_2$. Then, by Lemma B.4, we would have that $M_1^* = f_1^+(M^*) \geq f_1^-(M^*) = M_2^*$.

The inequality would be strict unless $M^* = M^\dagger$. However, the strict inequality contradicts the statement of Lemma B.3. On the other hand, equality also leads to a contradiction since it would yield $M_1^* = M_2^* > \frac{M^*}{2}$ (see Eq. (B.7)). Thus $\frac{2A_1}{1+\sqrt{1+\delta}} > \frac{2A_i}{1+\sqrt{1+\delta}}$ for all $i \geq 2$. Recalling the definition of $M^\dagger$ given in (B.6), this leads to $(2A_i + \delta M^\dagger)^2 - 4A_i^2(1+\delta) > (2A_i + \delta\frac{2A_i}{1+\sqrt{1+\delta}})^2 - 4A_i^2(1+\delta) = 0$ for $i \geq 2$. Moreover, it leads to $(2A_1 + \delta M^\dagger)^2 - 4A_1^2(1+\delta) = 0$. Next, we take the limit of the expression in (B.12) and obtain the result.

(P2) $\lim_{y\to M^\dagger} \frac{\partial m^+(y)}{\partial y} = \infty$.
Using the same arguments in (P1), the result follows directly after taking the limit of the expression in (B.11).

(P3) $\lim_{y\to\infty} \frac{f_1^+(y)}{y} = \frac{\delta}{1+\delta} < 1$ and $\lim_{y\to\infty} \frac{f_i^-(y)}{y} = 0$.
The first limit follows directly from the expression of f_1^+ given in Eq. (B.5). The second limit follows immediately upon rewriting Eq. (B.4) in the equivalent form

$$f_i^-(y) = \frac{A_i^2}{2A_i + \delta y + \sqrt{(2A_i+\delta y)^2 + 4A_i^2(\delta+1)}}.$$

(P4) $\lim_{y\to\infty} m^+(y) = -\infty$.
We can write $m^+(y) = y(-1 + \frac{f_1^+(y)}{y} + \frac{f_i^-(y)}{y})$, and note that by (P3), $\lim_{y\to\infty} -1 + \frac{f_1^+(y)}{y} + \frac{f_i^-(y)}{y} = -1 + \frac{\delta}{1+\delta} + 0 < 0$.

(P5) $m^+(A_1) > 0$. By definition of $m^+(y)$, we have that $m^+(A_1) = -A_1 + f_1^+(A_1) + \sum_{i=2}^I f_i^-(A_1) = -A_1 + A_1 + \sum_{i=2}^I f_i^-(A_1) > 0$.[11]

Combining (P1) and (P2) above, we deduce that the function $m^+(y)$ is concave and increasing at $M^\dagger$. The concavity of m^+ can only change if $\frac{\partial^2 m^+(y)}{\partial y^2} = 0$. Using Eq. (B.12), $\frac{\partial^2 m^+(y)}{\partial y^2} = 0$ if and only if $g(y) = 0$, where

$$\begin{aligned} g(y) :=& -A_1^2 + \sum_{i=2}^I \left(\frac{(2A_1+\delta y)^2 - 4A_1^2(1+\delta)}{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)}\right)^{3/2} A_i^2 \\ :=& -A_1^2 + \sum_{i=2}^I (h_i(y))^{3/2} A_i^2. \end{aligned}$$

Using straightforward algebraic manipulations, we obtain

$$\begin{aligned} h_i(y) &= \frac{(2A_1+\delta y)^2 - 4A_1^2(1+\delta)}{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)} \\ &= 1 + \frac{(2A_1+\delta y)^2 - 4A_1^2(1+\delta) - (2A_i+\delta y)^2 + 4A_i^2(1+\delta)}{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)} \\ &= 1 + \frac{4(A_1-A_i)(A_1+A_i+\delta y) - 4(A_1+A_i)(A_1-A_i)(1+\delta)}{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)} \\ &= 1 - 4\delta(A_1-A_i)\frac{(A_1+A_i)-y}{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)}. \end{aligned} \tag{B.13}$$

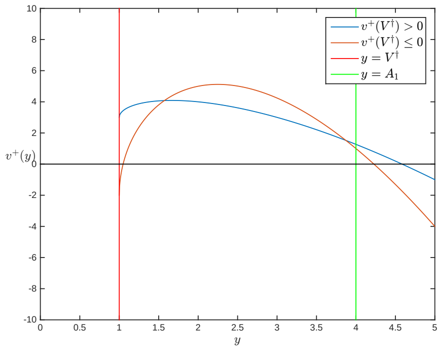
Notice that for $y \in [0, A_1 + A_i]$, $(A_1+A_i) - y$ is strictly decreasing in y and $(2A_i+\delta y)^2 - 4A_i^2(1+\delta)$ strictly increasing in y. Thus $\frac{(A_1+A_i)-y}{(2A_i+\delta y)^2-4A_i^2(1+\delta)}$ is strictly decreasing in y. Since $A_1 > A_i$, this implies that $h_i(y)$ is strictly increasing in y for $y \in [0, A_1 + A_i]$. Thus, $g(y)$ is increasing in y on the interval $[0, A_1 + \min_i\{A_i\}] = [0, A_1 + A_I] \supseteq [0, A_1]$. In this interval, there can be at most one point M_0 such that $g(M_0) = 0$. In other words, the concavity of $m^+(y)$ can change at most once on the interval $[0, A_1]$.

Now, we consider two cases.

[11] Recall that $f_i^-(y)$ is positive for all y since $2A_i + \delta y > \sqrt{(2A_i+\delta y)^2 - 4A_i^2(1+\delta)}$.

- **Case 1**: the concavity of $m^+(y)$ does not change on the interval $(M^\dagger, A_1]$. Then $m^+(M)$ is strictly concave on $(M^\dagger, A_1]$, since the function is concave and increasing near $M^\dagger$ by virtue of (P1) and (P2). Then, since $m^+(A_1) > 0$ by (P5), there must be a real value $M^* \in [M^\dagger, A_1]$ such that $m^+(M^*) = 0$ if $m^+(M^\dagger) \leq 0$. There can be no solution if $m^+(M^\dagger) > 0$. We refer the reader to figure 3 for a graphical illustration.

Figure 3: The case when $m^+(y)$ does not change its concavity in $(M^\dagger, A_1]$. There is exactly one solution if $m^+(M^\dagger) \leq 0$, and there is no solution if $m^+(M^\dagger) > 0$.



- **Case 2**: the concavity of $m^+(y)$ changes exactly once on the interval $(M^\dagger, A_1]$. Then there exists a real value $M_0 \in (M^\dagger, A_1]$ such that $g(M_0) = 0$ and $\frac{\partial^2 m^+(M)}{\partial M^2} = 0$. Hence, $m^+(y)$ is concave on $(M^\dagger, M_0]$ and convex on $[M_0, A_1]$.
  Recall that, for $i = 1, \ldots, I$, $h_i(y)$ is strictly increasing in y in the region $(M^\dagger, A_1 + A_i)$. This implies that $g(y) > g(M_0) = 0$ for all $y \in [M_0, A_1 + A_I]$. In addition, for $y > A_1 + A_i$, using the expression for h_i given in Eq. (B.13) we deduce

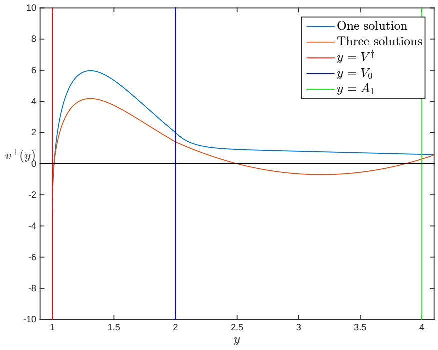
  $$h_i(y) = 1 - 4\delta(A_1 - A_i)\frac{(A_1 + A_i) - y}{(2A_i + \delta y)^2 - 4A_i^2(1+\delta)} > 1 = h_i(A_1 + A_i).$$

  Thus, for $y \in [A_1 + A_I, \infty)$

  $$\begin{aligned} g(y) &= -A_1^2 + \sum_{i=2}^{I} (h_i(y))^{3/2} A_i^2 \geq -A_1^2 + \sum_{i=2}^{I} \left(h_i(A_1 + \min_i A_i)\right)^{3/2} A_i^2 \\ &> -A_1^2 + \sum_{i=2}^{I} (h_i(A_1 + A_I))^{3/2} A_i^2 > -A_1^2 + \sum_{i=2}^{I} (h_i(M_0))^{3/2} A_i^2 = g(M_0) = 0. \end{aligned}$$

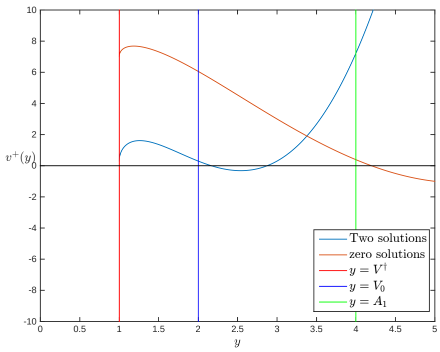
  Above, the second inequality follows from the fact that for each i, $h_i(y)$ is strictly increasing on $y \in [M_0, A_1 + A_I]$. Combining the analysis of the function $g(y)$ on both intervals $(M^\dagger, M_0]$ and $[M_0, A_1]$, we deduce that for $y \in (M^\dagger, \infty)$, $\frac{\partial^2 m^+(y)}{\partial y^2} = 0$ only at $y = M_0$.
  Next, we distinguish two subcases:
  - $m^+(M^\dagger) \leq 0$. Since $m^+(A_1) \geq 0$ by (P5), $m^+(y)$ is continuous, and the concavity of $m^+(y)$ changes exactly once, Eq. (B.10) can only have either one or three solutions. We rule out the case of three solutions using the following argument. Suppose by contradiction that there exist three solutions $s_1, s_2, s_3 \in [M^\dagger, A_1]$. Then it must be the case that $m^+(y) < 0$ in the interval (s_2, s_3). Thus $m^+(y)$ is not only convex but also increasing in the region $(s_3, A_1] \cap (M_0, A_1]$. Since the concavity of $m^+(y)$ cannot change in $[A_1, \infty)$, this contradicts (P4). We also refer the reader to figure 4 for a graphical illustration.

Figure 4: The case when $m^+(y)$ changes its concavity exactly once on $(M^\dagger, A_1]$ and $m^+(M^\dagger) \leq 0$. There can only be one solution to $m^+(y) = 0$. Existence of three solutions contradicts (P4).



– $m^+(M^\dagger) > 0$. Again, since the concavity of $m^+(y)$ changes exactly once, there can be either zero or two solutions to Eq. (B.10) in the interval $[M^\dagger, A_1]$. We rule out the case of two roots via the following argument. Suppose there exists two solutions $s_1, s_2 \in [M^\dagger, A_1]$. Then it must be the case that $m^+(y) < 0$ if $y \in (s_1, s_2)$, thus $m^+(y)$ is not only convex but also increasing in the region $(s_2, A_1] \cap (M_0, A_1]$. Since the concavity of $m^+(y)$ cannot change in $[A_1, \infty)$, this contradicts (P4). We also refer the reader to figure 5 for a graphical illustration.

Figure 5: The case when $m^+(y)$ changes its concavity exactly once in $(M^\dagger, A_1]$ and $m^+(M^\dagger) \leq 0$. There cannot be any solution of the equation $m^+(y) = 0$. The two solutions case contradicts (P4).



□

Step 4. The solution is unique.

We will show that class I and class II solutions are mutually exclusive, except for the case when they coincide. Notice that $m^+(M^\dagger) = m^-(M^\dagger)$. Hence, by lemmas B.5 and B.6, there exists one solution which is of class I if $m^+(M^\dagger) < 0$, and one solution which is of class II when $m^+(M^\dagger) > 0$. The case when $m^+(M^\dagger) = 0$ means that $M^\dagger$ is a solution which is both of class I and class II, because in this case $f_1^+(M^\dagger) = f_1^-(M^\dagger)$. In this case, the solution is still unique. □

Combining the proof of existence and uniqueness, we have completed the proof of Theorem 1. □

Proof of Proposition 2.1. Let $\pi_i := \theta\sqrt{1-\rho^2}(1-\kappa_i^*)$, we may rewrite Eq. (1.14) as

$$\begin{cases} dA_i = \phi A_i dt + \pi_i A_i dW_i \\ dA = \phi A dt + \sum_{i=1}^I \pi_i A_i dW_i \end{cases}.$$

Application of Itô's formula gives,

$$d\frac{A_i^2}{A^2} = 2\pi_i \frac{A_i^2}{A^2} dW_i - 2\frac{A_i^2}{A^3}\sum_{j=1}^I \pi_j A_j dW_j + \frac{A_i^2}{A^2}\left(\pi_i^2 - 4\frac{\pi_i}{A}\sum_{j=1}^I \pi_j A_j \Psi_{i,j} + 3\frac{1}{A^2}\sum_{j,k}^I \pi_j\pi_k A_j A_k \Psi_{j,k}\right) dt$$

$$= \frac{A_i^2}{A^2}\left(\pi_i - 2\frac{\sum_{j=1}^I \pi_j A_j \Psi_{i,j}}{A}\right)^2 dt + \frac{A_i^2}{A^4}\left(3\sum_{j,k}^I \pi_j\pi_k A_j A_k \Psi_{j,k} - 4\left(\sum_{j=1}^I \pi_j A_j \Psi_{i,j}\right)^2\right) dt$$

$$+ 2\pi_i \frac{A_i^2}{A^2} dW_i - 2\frac{A_i^2}{A^3}\sum_{j=1}^I \pi_j A_j dW_j.$$

Since $\pi_i \leq \theta\sqrt{1-\rho^2}$ and $A = \sum_{j=1}^I A_j$, all coefficients of the Itô integral terms are bounded. Thus all Itô integral terms are martingales. To show that $\frac{A_i^2}{A^2}$ is a submartingale, it suffices to show that the following inequality holds:

$$0 < 3\sum_{j,k}^I \pi_j\pi_k A_j A_k \Psi_{j,k} - 4\left(\sum_{j=1}^I \pi_j A_j \Psi_{i,j}\right)^2$$

$$= 3\sum_{j=1}^I \pi_j^2 A_j^2 + 3\sum_{k\neq j}^I \pi_j\pi_k A_j A_k \psi_j\psi_k - 4\pi_i^2 A_i^2 - 4\psi_i^2\sum_{j\neq i}\psi_j^2\pi_j^2 A_j^2 - 4\psi_i^2\sum_{j\neq k}^I \pi_j\pi_k A_j A_k \psi_j\psi_k.$$

Since $0 \leq \psi_i^2 \leq \frac{1}{\sqrt{2}} < \frac{\sqrt{3}}{2}$, we have

$$3\pi_j\pi_k A_j A_k \psi_j\psi_k - 4\psi_i^2 \pi_j\pi_k A_j A_k \psi_j\psi_k > 0.$$

Using the relation $\pi_i^2 A_i^2 = \sigma^2\frac{1-\rho^2}{\rho^2}\kappa_i^* A_i \sum_{j\neq i}\kappa_j^* A_j$ implied by Eq. (1.12) , we have

$$3\sum_{j=1}^I \pi_j^2 A_j^2 - 4\pi_i^2 A_i^2 - 4\psi_i^2\sum_{j\neq i}\psi_j^2\pi_j^2 A_j^2$$

$$= \sigma^2\frac{1-\rho^2}{\rho^2}\left(3\sum_{j\neq k}\kappa_j^*\kappa_k^* A_j A_k - 4A_i\kappa_i^*\sum_{j\neq i}\kappa_j^* A_j - 4\psi_i^2\sum_{j\neq i}\psi_j^2 A_j\kappa_j^*\sum_{k\neq j}\kappa_k^* A_k\right)$$

$$\geq \sigma^2\frac{1-\rho^2}{\rho^2}\left(3\sum_{j\neq k}\kappa_j^*\kappa_k^* A_j A_k - (4+\frac{4}{\sqrt{2}}\psi_i^2)A_i\kappa_i^*\sum_{j\neq i}\kappa_j^* A_j - 4\psi_i^2\sum_{j\neq k;j,k\neq i} A_j\kappa_j^*\kappa_k^* A_k\right)$$

$$= \sigma^2\frac{1-\rho^2}{\rho^2}\Bigg(3\sum_{j\neq k;j,k\neq i}\kappa_j^*\kappa_k^* A_j A_k - 4\psi_i^2\sum_{j\neq k;j,k\neq i}\kappa_j^*\kappa_k^* A_j A_k$$

$$+ 6A_i\kappa_i^*\sum_{j\neq i}\kappa_j^* A_j - (4+2\sqrt{2}\psi_i^2)A_i\kappa_i^*\sum_{j\neq i}\kappa_j^* A_j\Bigg) > 0.$$

Both of the above inequalities hold since $0 \leq \psi_i^2 \leq \frac{1}{\sqrt{2}}$ for all i. This shows that $\frac{A_i^2}{A^2}$ is a submartingale. Since η is a sum of submartingales, it is also a submartingale. ☐

Proof of Proposition 2.2. We start by proving the first statement. Using the functional forms of class I and class II solutions given in equations (B.4) and (B.5), we consider the following two cases.

Case 1: The unique solution is of class I or $i, j > 1$. Then, for $k = i, j$ we can write

$$\kappa_k^* = \frac{f^-(M)}{A_k} = \frac{2 + \delta\frac{M}{A_k} - \sqrt{(2 + \delta\frac{M}{A_k})^2 - 4(\delta+1)}}{2(1+\delta)} = \frac{2}{2 + \delta\frac{M}{A_k} + \sqrt{(2 + \delta\frac{M}{A_k})^2 - 4(\delta+1)}},$$

which is clearly increasing in A_k. Thus $\kappa_i^* \geq \kappa_j^*$ if and only if $A_i \geq A_j$.

Case 2: The unique solution is of class II and $i = 1$, $j > 1$. Without loss of generality, let $A_1 = \max_i A_i$. Then

$$\kappa_1^* = \frac{f^+(M)}{A_1} = \frac{2 + \delta\frac{M}{A_1} + \sqrt{(2 + \delta\frac{M}{A_1})^2 - 4(\delta+1)}}{2(1+\delta)} \geq \frac{2 + \delta\frac{M}{A_1} - \sqrt{(2 + \delta\frac{M}{A_1})^2 - 4(\delta+1)}}{2(1+\delta)}$$
$$\geq \frac{2 + \delta\frac{M}{A_j} - \sqrt{(2 + \delta\frac{M}{A_j})^2 - 4(\delta+1)}}{2(1+\delta)} = \kappa_j^*.$$

Thus $\kappa_1^* \geq \kappa_j^*$ since we have $A_1 > A_j$.

Next, we prove the second statement. If $A_i > A_j$, using the result proven above, we have

$$\frac{M_i - M_j}{A_i - A_j} = \frac{\kappa_i^* A_i - \kappa_j^* A_j}{A_i - A_j} > \frac{\kappa_i^* A_i - \kappa_i^* A_j}{A_i - A_j} = \kappa_i^*,$$

and

$$\frac{M_i - M_j}{A_i - A_j} = \frac{\kappa_i^* A_i - \kappa_j^* A_j}{A_i - A_j} > \frac{\kappa_j^* A_i - \kappa_j^* A_j}{A_i - A_j} = \kappa_j^*.$$

This concludes the proof of the proposition. □

Proof of Proposition 2.3. We consider the two cases separately.

- $\delta \neq 1$. Solving Eq. (1.12) using the quadratic formula, we obtain

$$\kappa_1^* = \frac{\delta - \delta\frac{A_2}{A_1} - 2 \pm \sqrt{\left(\delta - \delta\frac{A_2}{A_1} - 2\right)^2 + 4(\delta - 1)}}{2(\delta-1)},$$
$$\kappa_2^* = \frac{\delta - \delta\frac{A_1}{A_2} - 2 \pm \sqrt{\left(\delta - \delta\frac{A_1}{A_2} - 2\right)^2 + 4(\delta - 1)}}{2(\delta-1)}.$$

In order to have $0 \leq \kappa_1^*, \kappa_2^* \leq 1$, we must either take the root of each equation with the plus sign, or with the minus sign. We now show that the roots with the plus signs, denoted by κ_1^+, κ_2^+, are both in $[0, 1]$.

If $\delta - \delta\frac{A_2}{A_1} - 2 \geq 0$, we must have $\delta \geq 2$. Then

$$\kappa_1^+ \leq \frac{\delta - 2 + \sqrt{(\delta-2)^2 + 4(\delta-1)}}{2(\delta-1)} = 1.$$

If $\delta - \delta\frac{A_2}{A_1} - 2 < 0$, then

$$\begin{aligned}\kappa_1^+ &= \frac{4(\delta-1)}{2(\delta-1)\left(\left|\delta - \delta\frac{A_2}{A_1} - 2\right| + \sqrt{\left(\delta - \delta\frac{A_2}{A_1} - 2\right)^2 + 4(\delta-1)}\right)} \\ &= \frac{2}{x + \sqrt{x^2 + 4(\delta-1)}},\end{aligned} \tag{B.14}$$

where $x = \left|\delta - \delta\frac{A_2}{A_1} - 2\right| = -\delta + \delta\frac{A_2}{A_1} + 2 \geq 0$. Set $y = \frac{A_2}{A_1}$. Differentiating $x + \sqrt{x^2 + 4(\delta-1)}$ with respect to y we have:

$$\frac{\partial(x + \sqrt{x^2 + 4(\delta - 1)})}{\partial y} = \delta + \frac{x\delta}{\sqrt{x^2 + 4(\delta - 1)}} > 0.$$

Next, view $\kappa_1^+ := \kappa_1^+(y)$ as a function of y. Since the derivative is always nonzero, the extrema of $\kappa_1^+(y)$ can only occur at the boundaries when $y = 0$, $y = \infty$, or $\delta - \delta y - 2 = 0$ (that is, $y = \frac{\delta - 2}{\delta}$). Notice that the last case is only relevant when $\delta \geq 2$. We have

$$\kappa_1^+(y)|_{y=0} = \frac{2}{(-\delta + 2) + \sqrt{(\delta - 2)^2 + 4(\delta - 1)}} = 1.$$
$$\kappa_1^+(y)|_{y=\infty} = 0.$$
$$\kappa_1^+(y)|_{y=\frac{\delta-2}{\delta}} = \frac{2}{\sqrt{4(\delta - 1)}} = \frac{1}{\sqrt{\delta - 1}} \leq 1.$$

The last inequality holds since we are considering the case where $\delta \geq 2$. Thus, $0 \leq \kappa_1^* \leq 1$. An analogous argument shows that $0 \leq \kappa_2^* \leq 1$. Since (κ_1^*, κ_2^*) is a solution and the solution is unique, this means that the solution is given by (κ_1^+, κ_2^+). The monotonicity statement follows immediately by differentiating the expressions of κ_1^+ and κ_2^+ with respect to $\frac{A_1}{A_2}$.

- $\delta = 1$. Solving the hedge equations in (1.12) directly gives the first equality in (2.4) and (2.5). Next, we prove the second equality. Notice that when $\delta < 2$, we have that $\delta - \delta\frac{A_2}{A_1} - 2 < 0$. Using Eq. (B.14), we then have

$$\lim_{\delta \to 1} \kappa_1^+(\delta) = \frac{2}{2\frac{A_2}{A_1} + 2} = \frac{A_1}{A_1 + A_2}.$$

A similar argument can be used to prove the second equality in (2.5).

□

Proof of Proposition 2.4. Applying Itô's formula as in the proof of Proposition 2.1, we have

$$d\eta = d\sum_{i=1}^{I} \frac{A_i^2}{A^2} = 2U \sum_{i=1}^{I} \left(\frac{A_i}{A^2} dt - I\frac{A_i^2}{A^3} dt \right) + \sum_{i=1}^{I} \Bigg(2\pi_i \frac{A_i^2}{A^2} dW_i^d - 2\frac{A_i^2}{A^3} \sum_{j=1}^{I} \pi_j A_j dW_j^d + \pi_i^2 \frac{A_i^2}{A^2} dt \\ - 4\pi_i \frac{A_i^2}{A^3} \sum_{j=1}^{I} \pi_j A_j \psi_{i,j} dt + 3\frac{A_i^2}{A^4} \sum_{j,k}^{I} \pi_j \pi_k A_j A_k \psi_{j,k} dt \Bigg). \tag{B.15}$$

Notice that only the first term in Eq (B.15) depends on U. It holds that

$$U \sum_{i=1}^{I} \left(\frac{A_i}{A^2} - I\frac{A_i^2}{A^3} \right) = \frac{U}{A^3} \sum_{i=1}^{I} \left(A_i A - I A_i^2 \right) = \frac{U}{A}(1 - I\eta).$$

Since $\eta(t_0) > \frac{1}{I}$, for any bounded continuous process U satisfying

$$U(t_0) \geq \frac{A\Bigg(\pi_i^2 \frac{A_i^2}{A^2} - 4\pi_i \frac{A_i^2}{A^3} \sum_{j=1}^{I} \pi_j A_j \psi_{i,j} + 3\frac{A_i^2}{A^4} \sum_{j,k}^{I} \pi_j \pi_k A_j A_k \psi_{j,k} \Bigg)}{I\eta - 1} \Bigg|_{t_0},$$

we obtain the inequality in (2.6). □

Proof of Proposition 3.1. Proceeding as in the proof of Proposition 2.1 and using Eq. (3.4), we have

$$\begin{cases} dA_i = \phi A_i dt + \pi_i A_i dW_i - \mu \left(\kappa_i^* A_i - \frac{M}{I} \right) dt, \\ dA = \phi A dt + \sum_{i=1}^{I} \pi_i A_i dW_i. \end{cases}$$

Then

$$d\frac{A_i^2}{A^2} = 2\pi_i\frac{A_i^2}{A^2}dW_i - 2\frac{A_i^2}{A^3}\sum_{j=1}^{I}\pi_j A_j dW_j + \pi_i^2\frac{A_i^2}{A^2}dt$$
$$- 4\pi_i\frac{A_i^2}{A^3}\sum_{j=1}^{I}\pi_j A_j\psi_{i,j}dt + 3\frac{A_i^2}{A^4}\sum_{j,k}^{I}\pi_j\pi_k A_j A_k\psi_{j,k}dt$$
$$- 2\frac{A_i}{A^2}\mu\left(\kappa_i^* A_i - \frac{M}{I}\right)dt,$$

and

$$\begin{aligned} d\eta &= d\sum_{i=1}^{I}\frac{A_i^2}{A^2} \\ &= \sum_{i=1}^{I}\Bigg(2\pi_i\frac{A_i^2}{A^2}dW_i - 2\frac{A_i^2}{A^3}\sum_{j=1}^{I}\pi_j A_j dW_j + \pi_i^2\frac{A_i^2}{A^2}dt \\ &\quad - 4\pi_i\frac{A_i^2}{A^3}\sum_{j=1}^{I}\pi_j A_j\psi_{i,j}dt + 3\frac{A_i^2}{A^4}\sum_{j,k}^{I}\pi_j\pi_k A_j A_k\psi_{j,k}dt\Bigg) \\ &\quad - 2\frac{\mu}{A^2}\sum_{i=1}^{I}\left(\kappa_i^* A_i - \frac{M}{I}\right)A_i dt. \end{aligned} \tag{B.16}$$

Above, all coefficients of the Itô integral terms are bounded and thus the Itô integral terms are martingales. By Proposition 2.2, the quantity $\sum_{i=1}^{I}(\kappa_i^* A_i - \frac{M}{I})A_i$ is positive as long as not all firm values are equal (the fully diversified case). This can be easily seen by viewing $\frac{1}{I}\sum_{i=1}^{I}(\kappa_i^* A_i - \frac{M}{I})A_i = \frac{1}{I}\sum_{i=1}^{I}(\kappa_i^* A_i - \overline{\kappa^* A})A_i$ as a sample covariance estimate. Here, $\overline{\kappa^* A}$ is the sample mean of the set $\{\kappa_i^* A_i\}_{i=1}^{I}$. Since $\kappa_i^* A_i$ is increasing in A_i by Lemma B.3, the sample covariance estimate must be positive. In addition, the terms in the first summation in Eq. (B.16) are independent of μ. Thus, if μ is a bounded continuous process such that

$$\mu(t_0) > \left.\frac{\left(\pi_i^2 A_i^2 - 4\pi_i\frac{A_i^2}{A}\sum_{j=1}^{I}\pi_j A_j\psi_{i,j} + 3\frac{A_i^2}{A^2}\sum_{j,k}^{I}\pi_j\pi_k A_j A_k\psi_{j,k}\right)}{2\sum_{i=1}^{I}\left(\kappa_i^* A_i - \frac{M}{I}\right)A_i}\right|_{t_0},$$

we have the inequality in (3.5) □

Proof of Proposition A.1. First we show that for all $x \in \mathbb{R}_+^I$, $\mathbf{\Sigma}(x)$ is positive semidefinite, by showing that all the principal minors are nonnegative. Recall, $\tilde{x} := \max_i x_i$ and $\bar{x} := \sum_{i=1}^{I} x_i$. The statement is obvious if $\tilde{x} = \bar{x}$. Now suppose $\tilde{x} < \bar{x}$. Consider any $m \times m$ principal submatrix of $\mathbf{\Sigma}(x)$. Due to notational symmetry, we can assume without loss of generality that it is the leading $m \times m$ principal submatrix $\mathbf{\Sigma}_m(x)$. Based on straightforward row and column operations, we have:

$$\det(\mathbf{\Sigma}_m(x)) = \bar{x}^{m-1}\sum_{n=m+1}^{I} x_n\prod_{n=1}^{m}(\bar{x} - x_n)^{-1} \geq 0.$$

Moreover, by setting $m = I$, we see that $\det(\mathbf{\Sigma}(x)) = 0$. Since $\mathbf{\Sigma}(x)$ is positive semidefinite, there exists a unique positive semidefinite matrix $\mathbf{\Gamma}(x)$ such that $\mathbf{\Sigma}(x) = \mathbf{\Gamma}^2(x)$, its principal square root.

Let

$$\mathbf{w}(x) := (w_1(x), \ldots, w_I(x)), \tag{B.17}$$

where $w_i(x) := \sqrt{x_i(\bar{x} - x_i)}$ and define Z as

$$Z(t) := \int_0^t \sum_{i=1}^{I}\sqrt{X_i(s)(\bar{X}(s) - X_i(s))}dW_i^{\mathbf{\Sigma}}(s) = \int_0^t \mathbf{w}(\mathbf{X}(s))\mathbf{\Gamma}(\mathbf{X}(s))d\mathbf{B}(s), \tag{B.18}$$

where $\mathbf{B}(s)$ is $I-$dimensional standard Brownian motion. If $\bar{X}(t) = \tilde{X}(t)$, we have $L(X(t)) = 0$ and $Z(t) = 0$. Otherwise, we observe that $Z(t)$ has zero quadratic variation:

$$d[Z, Z](t) = \left(\sum_{i=1}^{I} w_i^2(X(t)) - \sum_{i \neq j}^{I} X_i(t) X_j(t) \right) dt = 0.$$

Since Z has the representation given by Eq. (B.18), we can also write

$$[Z, Z](t) = \int_0^t (\mathbf{w}(s)\mathbf{\Gamma}(s))(\mathbf{w}(s)\mathbf{\Gamma}(s))^T ds$$

Therefore having zero quadratic variation means $\mathbf{w}(s)\mathbf{\Gamma}(s) = 0$ almost everywhere, which means $Z(t) = 0$. This completes the proof of the statement.

□

Proof of Proposition A.2. It is well known that the I-allele Wright-Fisher diffusion process is uniquely characterized by

1. $p_I(t) = 1 - \sum_{i=1}^{I-1} p_i(t)$ for all t.
2. For any $f \in C^2(\Delta_{I-1})$, $p(t)$ is the unique solution to the martingale problem

$$f(p(t)) - \int_0^t \mathcal{L}f(p(s))\, ds \quad \text{is a martingale,}$$

$$\mathcal{L}f(x) = \frac{1}{2} \sum_{i,j=1}^{I-1} x_i(\delta_{i,j} - x_j).$$

Now suppose $\mathbf{X}(t)$ is a weak solution to the SDE

$$\begin{cases} d\mathbf{X}(t) = \sqrt{\mathbf{X}(t) \circ (\bar{X}(t)\mathbf{1} - \mathbf{X}(t))} \circ \mathbf{\Gamma}(\mathbf{X}(t)) d\mathbf{B}(t) \\ \mathbf{X}(0) = p_0 \end{cases} . \tag{B.19}$$

Thus $\sum_{i=1}^{I} X_i(t)$ is constant by proposition A.1. Since $\sum_{i=1}^{I} X_i(0) = \sum_{i=1}^{I} p_{0,i} = 1$, $\sum_{i=1}^{I} X_i(t) = 1$ for all t. In other words, we can rewrite Eq. (B.19) as

$$\begin{cases} dX_i(t) = \sqrt{X_i(t)(1 - X_i(t))} \sum_{k=1}^{I} \mathbf{\Gamma}_{i,k} dB_k(t),\ i = 1 \ldots, I-1 \\ X_I(t) = 1 - \sum_{i=1}^{I-1} X_i(t) \\ \mathbf{X}(0) = p_0 \end{cases} . \tag{B.20}$$

Notice that the associated infinitesimal generator for $f \in C^2(\Delta_{I-1})$ is

$$\begin{aligned} \mathcal{L}_{\mathbf{X}} f(\mathbf{x}) &= \frac{1}{2} \sum_{i,j=1}^{I-1} \sum_{k=1}^{I} \sqrt{x_j(1 - x_j) x_i(1 - x_i)} \mathbf{\Gamma}_{i,k} \mathbf{\Gamma}_{j,k} \frac{\partial^2 f}{\partial_i \partial_j}(x) \\ &= \frac{1}{2} \sum_{i,j=1}^{I-1} \sqrt{x_j(1 - x_j) x_i(1 - x_i)} \mathbf{\Sigma}_{i,j} \frac{\partial^2 f}{\partial_i \partial_j}(x) \\ &= \frac{1}{2} \sum_{i,j=1}^{I-1} x_i(\delta_{i,j} - x_j) \frac{\partial^2 f}{\partial_i \partial_j}(x), \end{aligned}$$

which is the same as the infinitesimal generator defining the I-allele Wright-Fisher diffusion process. Thus, $p(t)$ is the unique weak solution to Eq. (B.20). Thus it is also the unique weak solution to the equivalent formulation Eq. (A.4). □

Proof of Proposition A.3. Using the same notation as in the proof of Proposition A.2, write

$$\mathbf{Z}^{\mathbf{\Sigma}}_{n+1} = \mathbf{\Gamma}(\mathbf{X}(t_m))\mathbf{D}_{n+1}$$
$$\mathbf{D}_n|\mathcal{F}_{t_m} \sim \mathcal{N}(0,_I \Delta t)$$

Define

$$U(t_{m+1}) := \sum_{i=1}^{I} \sqrt{X_i(t_m)(\bar{X}(t_m) - X_i(t_m))} Z^{\mathbf{\Sigma}}_{i,n+1} = \mathbf{w}(\mathbf{X}(t_m))\mathbf{\Gamma}(\mathbf{X}(t_m))\mathbf{D}_{n+1}.$$

Then

$$\begin{aligned}\mathbb{E}[U^2(t_{m+1})|\mathcal{F}_{t_m}] &= \mathbb{E}[\mathbf{w}(\mathbf{X}(t_m))\mathbf{\Gamma}(\mathbf{X}(t_m))\mathbf{D}_{n+1}\mathbf{D}^T_{n+1}\mathbf{\Gamma}(\mathbf{X}(t_m))\mathbf{w}^T(\mathbf{X}(t_m))|\mathcal{F}_{t_m}] \\ &= \mathbf{w}(\mathbf{X}(t_m))\mathbf{\Sigma}(\mathbf{X}(t_m))\mathbf{w}^T(\mathbf{X}(t_m))\Delta t = 0,\end{aligned}$$

where we have used that $\mathbf{\Gamma}$ is symmetric. The last equality follows by a direct calculation using the expression for $\mathbf{w}$ given in Eq. (B.17) and of $\mathbf{\Sigma}$ given in Eq. (A.2). Since $U(t_{m+1})$ has zero variance, it must be constant, which means that $\mathbf{w}(\mathbf{X}(t_m))\mathbf{\Gamma}(\mathbf{X}(t_m))\mathbf{D}_{n+1}$ is constant. Since the components of $\mathbf{D}_{n+1}$ are independent, this means that $\mathbf{w}(\mathbf{X}(t_m))\mathbf{\Gamma}(\mathbf{X}(t_m)) = 0$ and $U(t_{m+1}) = 0$. □

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